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The hybrid Euler–Lagrange procedure using an extension of Moffatt's method

A. M. SOWARD[1]† AND P. H. ROBERTS[2]

[1]School of Mathematical Sciences, University of Exeter, Exeter EX4 4QE, UK

[2]Department of Mathematics, University of California, Los Angeles, CA 90095, USA

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The hybrid Euler–Lagrange (HEL) description of fluid mechanics, pioneered largely by Andrews & McIntyre (*J. Fluid Mech.*, vol. 89, 1978, pp. 609–646), has had to face the fact, in common with all Lagrangian descriptions of fluid motion, that the variables used do not describe conditions at the coordinate $\boldsymbol{x}$, upon which they depend, but conditions elsewhere at some displaced position $\boldsymbol{x}^L(\boldsymbol{x},t)=\boldsymbol{x}+\boldsymbol{\xi}(\boldsymbol{x},t)$, generally dependent on time t. To address this issue, we employ 'Lie dragging' techniques of general tensor calculus to extend a method introduced by Moffatt (*J. Fluid Mech.*, vol. 166, 1986, pp. 359–378) in the fluid dynamic context, whereby the point $\boldsymbol{x}$ is dragged to $\boldsymbol{x}^L(\boldsymbol{x},t)$ by a 'fictitious steady flow' $\boldsymbol{\eta}^*(\boldsymbol{x},t)$ in a unit of 'fictitious time'. Whereas $\boldsymbol{\xi}(\boldsymbol{x},t)$ is a Lagrangian concept intimately linked to the location $\boldsymbol{x}^L(\boldsymbol{x},t)$, the 'dragging velocity' $\boldsymbol{\eta}^*(\boldsymbol{x},t)$ has an essentially Eulerian character, because it describes the fictitious velocity at $\boldsymbol{x}$ itself. For the case of constant-density fluids, we show, using solenoidal $\boldsymbol{\eta}^*(\boldsymbol{x},t)$ instead of solenoidal $\boldsymbol{\xi}(\boldsymbol{x},t)$, how the HEL theory can be cast into Eulerian form. A useful aspect of this Eulerian development is that the mean flow itself remains solenoidal, a feature that traditional HEL theories lack. Our method realizes the objective sought by Holm (*Physica* D, vol. 170, 2002, pp. 253–286) in his derivation of the Navier–Stokes–α equation, which is the basis of one of the methods currently employed to represent the sub-grid scales in large-eddy simulations. His derivation, based on expansion to second order in $\boldsymbol{\xi}$, contained an error which, when corrected, implied a violation of Kelvin's theorem on the constancy of circulation in inviscid incompressible fluid. We show that this is rectified when the expansion is in $\boldsymbol{\eta}^*$ rather than $\boldsymbol{\xi}$, Kelvin's theorem then being satisfied to all orders for which the expansion converges. We discuss the implications of our approach using $\boldsymbol{\eta}^*$ for the Navier–Stokes–α theory.

Key words: Navier–Stokes equations, turbulence theory

1. Introduction

Three decades ago Andrews & McIntyre (1978*a,b*) found a significant way of representing the effects of wave motion on a mean flow. They used a hybrid Euler–Lagrange (HEL) representation, so called because it is a compromise between the familiar Eulerian and Lagrangian representations. Previously, the HEL method had been employed by Soward (1972) in the geodynamo context to incorporate the effect of the non-axisymmetric components of a magnetic field on the generation of its

† Email address for correspondence: a.m.soward@exeter.ac.uk

axisymmetric part. The introduction of this method into fluid mechanics is often attributed to Eckart (1963), although some of the essential ideas had previously been advanced by Frieman & Rotenberg (1960). It is basic to what follows.

A brief exposition of the HEL technique is the subject of § 2, where the HEL form of the Euler equations,

$$\mathscr{D}_t \boldsymbol{v}^* = -\nabla(p^*/\rho_0), \quad \nabla \cdot \boldsymbol{v}^* = 0, \tag{1.1a,b}$$

governing the velocity $\boldsymbol{v}^*(\boldsymbol{x}, t)$ of an inviscid fluid of uniform density ρ_0 at position $\boldsymbol{x}$ and time t is derived; here only a sketch of the derivation is given. In (1.1a,b), $p^*(\boldsymbol{x}, t)$ is pressure,

$$\mathscr{D}_t \equiv \partial_t + \boldsymbol{v}^* \cdot \nabla \tag{1.1c}$$

is the material derivative and $\partial_t \equiv \partial/\partial t$. A more convenient form of (1.1a) for what follows is

$$\partial_t \boldsymbol{v}^* + \boldsymbol{v}^* \cdot \nabla \boldsymbol{v}^* + \nabla\left(\tfrac{1}{2}|\boldsymbol{v}^*|^2\right) = -\nabla \Pi^*, \tag{1.1d}$$

where

$$\Pi^* \equiv (p^*/\rho_0) - \tfrac{1}{2}|\boldsymbol{v}^*|^2 \tag{1.1e}$$

is a modified pressure. This form arises naturally when Euler's equation is derived from Hamilton's principle (see e.g. Soward & Roberts 2008, equation (2.23b)). The minus (as opposed to plus) sign multiplying $|\boldsymbol{v}^*|^2/2$ in the definition (1.1e) of Π^* may be unexpected. The point is that the combination $\boldsymbol{v}^* \cdot \nabla \boldsymbol{v}^* + \nabla(|\boldsymbol{v}^*|^2/2)\,(=\{\boldsymbol{v}^*, \boldsymbol{v}^*\}$, see below) is needed on the left-hand side of (1.1d) to generate the invariant structure (1.3b) that emerges in the transformed equation (1.3a). An important property of both (1.1a) and (1.1d) is that they imply Kelvin's theorem that the circulation round any closed curve $\mathscr{C}^*(t)$ that moves with the fluid is constant:

$$\frac{\mathrm{d}}{\mathrm{d}t} \oint_{\mathscr{C}^*} \boldsymbol{v}^*(\boldsymbol{x}, t) \cdot \mathrm{d}\boldsymbol{x} = 0. \tag{1.1f}$$

Physical variables in the fluid are identified by the superscript *.

The HEL transformation of (1.1a,b) is based on a mapping,

$$\boldsymbol{x} \mapsto \boldsymbol{x}^L(\boldsymbol{x}, t) \equiv \boldsymbol{x} + \boldsymbol{\xi}(\boldsymbol{x}, t), \tag{1.2}$$

of a reference position $P : \boldsymbol{x}$ with Cartesian coordinates x_i to another displaced HEL position P^L: $\boldsymbol{x}^L \equiv (x_1^L, x_2^L, x_3^L)$, at which the HEL formulation of the governing equations applies. When the reference position P is stationary, the position P^L moves with velocity $\partial_t \boldsymbol{x}^L$. On the other hand, when P^L is a material point that moves with the fluid, advection relative to that moving point is measured by $\boldsymbol{u}^*(\boldsymbol{x}^L, t) \equiv \boldsymbol{v}^*(\boldsymbol{x}^L, t) - \partial_t \boldsymbol{x}^L$. Consider now the physical thought-experiment whereby the moving fluid particle at P^L is dragged back to P instantaneously but with the subsequent motion of P tied to the motion of P^L. Then the velocity $\boldsymbol{u}(\boldsymbol{x}, t)$ of P is related to advection velocity $\boldsymbol{u}^*(\boldsymbol{x}^L, t)$, i.e. the true velocity $\boldsymbol{v}^*(\boldsymbol{x}^L, t)$ of P^L with $\partial_t \boldsymbol{x}^L$ removed, by $\boldsymbol{u}(\boldsymbol{x}, t) \equiv \boldsymbol{u}^*(\boldsymbol{x}^L, t) \cdot (\nabla \boldsymbol{x}^L)^{-1}$. This physical picture is relatively straightforward and confirmed from a more formal point of view by (2.8a–c) and (2.9a,b) below. The key idea, which HEL builds upon, is that the motion of P is relatively smooth, and constitutes the primary motion, such as a mean flow. Any secondary motion, such as waves or some other fluctuating flow, which rides on the primary flow, is accommodated by the movement of the true position P^L relative to P (Soward 1972).

Whereas the velocity $\boldsymbol{u}(\boldsymbol{x},t)\equiv\boldsymbol{u}^*(\boldsymbol{x}^L,t)\cdot(\nabla\boldsymbol{x}^L)^{-1}$ describes advection, the velocity $\boldsymbol{V}(\boldsymbol{x},t)\equiv(\nabla\boldsymbol{x}^L)\cdot\boldsymbol{v}^*(\boldsymbol{x}^L,t)$ provides a measure of momentum, as the HEL transformation (see (1.3) below) of Euler's equation reveals. Respectively, the velocities $\boldsymbol{V}$ and $\boldsymbol{u}$ have the general tensor interpretations of covariant and contravariant vectors as explained in Appendix A. This interpretation, however, requires the Cartesian coordinates x_i of $\boldsymbol{x}$ to be non-Cartesian coordinates of the position P^L. This conceptual step is a well-known general tensor device employed when a tensor is 'Lie dragged' from P to P^L. In our development, we will (except in Appendix A) take the Cartesian point of view encapsulated by (1.2). It means that we have not created a true general tensor environment, which ensues only after the coordinates x_i are given the non-Cartesian interpretation of defining the position P^L rather than P. Despite this caveat, we will retain general tensor terminology, like covariant and contravariant, to identify the general tensor equivalent, even though the terms 'quasi-covariant' and 'quasi-contravariant' might convey the spirit of our usage better. Henceforth, we will refer to all dependent variables, such as $\boldsymbol{V}$ and $\boldsymbol{u}$ at the reference position $P:\boldsymbol{x}\equiv(x_1,x_2,x_3)$, which define flow properties at P^L, constructed by the tensorial rules as HEL variables, and to x_i as the HEL coordinates. This is the HEL formulation, for which the superscript * appearing on the original physical variables is omitted.

We show in §2 that the HEL transformation of Euler's equation (1.1*d*) at $P^L:\boldsymbol{x}^L$ leads to the covariant HEL form

$$\partial_t\boldsymbol{V}+\mathrm{L}_{\boldsymbol{u}}\boldsymbol{V}=-\nabla\Pi, \tag{1.3a}$$

at the reference position $P:\boldsymbol{x}$, where

$$\mathrm{L}_{\boldsymbol{u}}\boldsymbol{V}\equiv\{\boldsymbol{u},\boldsymbol{V}\}\equiv\boldsymbol{u}\cdot\nabla\boldsymbol{V}+(\nabla\boldsymbol{u})\cdot\boldsymbol{V} \tag{1.3b}$$

is the Lie derivative of a covariant vector, whose significance becomes clear in §3. A loose interpretation of (1.3*a*) is that $\boldsymbol{V}(\boldsymbol{x},t)$ represents the momentum per unit mass of the flow advected by $\boldsymbol{u}(\boldsymbol{x},t)$. Since (1.3*a*) is an exact consequence of (1.1*d*), it, not unexpectedly, also yields a form of Kelvin's theorem

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\mathscr{C}}\boldsymbol{V}\cdot\mathrm{d}\boldsymbol{x}=0, \tag{1.3c}$$

where $\mathscr{C}(t)\mapsto\mathscr{C}^*(t)$ under (1.2) and $\mathscr{C}(t)$ is advected by the flow $\boldsymbol{u}(\boldsymbol{x},t)$.

Andrews & McIntyre (1978*a*,*b*) showed that the HEL form (1.3*a*) is uniquely suited to the investigation of waves riding on a mean flow. We, however, stress that Andrews & McIntyre's original work and many subsequent developments, including Bühler & McIntyre (2003, 2005), have focused on geophysical flows that include other ingredients such as rotation and compressibility (see also Bühler 2009). In order to present our ideas in the simplest possible way we ignore these other processes here and restrict attention to Euler's equation (1.3*a*) for an incompressible flow. Andrews & McIntyre's essential idea was that the wave motion is captured by the Lagrangian displacement $\boldsymbol{\xi}$, whereas the advective velocity $\boldsymbol{u}$ in the reference frame has no wave contribution, i.e. $\overline{\boldsymbol{\xi}}=\boldsymbol{0}$, $\boldsymbol{u}=\overline{\boldsymbol{u}}$, where the bar denotes the average. Under these assumptions they showed that $\boldsymbol{u}$ is simply the Lagrangian average $\overline{\boldsymbol{v}^{*L}}$ of the flow velocity $\boldsymbol{v}^*(\boldsymbol{x}^L,t)$ in a sense that will be made clear in §4.1 (see (4.5*d*)). The mean of (1.3*a*) is then simply

$$\partial_t\overline{\boldsymbol{V}}+\mathrm{L}_{\boldsymbol{u}}\overline{\boldsymbol{V}}=-\nabla\overline{\Pi}. \tag{1.4a}$$

The absence of any contribution from the fluctuating part of $\boldsymbol{V}$ in this equation is striking and is the *raison d'être* for what Andrews & McIntyre (1978*a*) called the generalized Lagrangian mean (GLM) theory. A further consequence of (1.4*a*) is that Kelvin's theorem again applies, this time to the circulation of the mean velocity $\overline{\boldsymbol{V}}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\mathscr{C}} \overline{\boldsymbol{V}} \cdot \mathrm{d}\boldsymbol{x} = 0. \tag{1.4b}$$

This is simply a special case that omits the aforementioned geophysical ingredients of Andrews & McIntyre (1978*a*, equation (3.11)); this formula is expressed in terms of their pseudomomentum $\mathbf{p} = \overline{\boldsymbol{v}^{*L}} - \overline{\boldsymbol{V}}$ (see also § 4.1). Indeed, their identification of the pseudomomentum $\mathbf{p}$, as a natural variable in the HEL approach, was a considerable triumph and a main reason for the continuing interest in the subject.

The attraction of being able to preserve in the GLM theory a property of the inviscid equations as basic as Kelvin's theorem led Holm (2002) to construct a derivation of the Navier–Stokes–alpha (NS–α) equations guided by GLM considerations. The NS–α equations are a modification of the Navier–Stokes equations that has proved useful in securing numerical convergence in large-eddy simulations (see Geurts, Kuczag & Titi 2008 and references therein). Holm recognized that such a derivation would have to surmount the difficulty mentioned in the abstract: the GLM theory describes the flow at $\boldsymbol{x}^L$ in terms of the coordinates of a different point, namely the mean position $\boldsymbol{x} = \overline{\boldsymbol{x}^L}$ (since $\overline{\boldsymbol{\xi}} = \boldsymbol{0}$). He saw that it is necessary to transform from the HEL to the Eulerian description, but this transformation is far from straightforward except for small $\boldsymbol{\xi}$. In that case, a connection can be made by use of Taylor series expansions that relate conditions at $\boldsymbol{x}^L$ to those at $\boldsymbol{x}$. Then, under the assumptions $\overline{\boldsymbol{\xi}} = \boldsymbol{0}$ and $\boldsymbol{u} = \overline{\boldsymbol{u}}$, the mean and fluctuating parts of the Eulerian velocity, expressed in terms of HEL variables, are

$$\overline{\boldsymbol{v}^*} = \boldsymbol{u} + O(\xi^2) \quad \text{and} \quad \boldsymbol{v}^{*\prime} = \partial_t \boldsymbol{\xi} + \boldsymbol{u} \cdot \nabla \boldsymbol{\xi} - \boldsymbol{\xi} \cdot \nabla \boldsymbol{u} + O(\xi^2), \tag{1.5a,b}$$

respectively. The latter expression for $\boldsymbol{v}^{*\prime}$ is close to a standard exact representation for the Eulerian fluctuating velocity, in which $\boldsymbol{u}$ is replaced by $\overline{\boldsymbol{v}^*}$ and $\boldsymbol{\xi}$ is replaced by another displacement vector that differs from it only by $O(\xi^2)$. Use of this exact Eulerian representation led Holm to what he called 'glm theory' to contrast it with the GLM theory.

To complete the glm method and set up a deterministic system governing the evolution of the Eulerian mean flow $\overline{\boldsymbol{v}^*}$, a closure assumption is needed. Holm accomplishes this by introducing what he called a Taylor hypothesis, which leads to the NS–α equations involving both $\overline{\boldsymbol{v}^*}$ (loosely linked to $\boldsymbol{u}$ of the GLM theory) and a second mean velocity (loosely linked to $\overline{\boldsymbol{V}}$ of the GLM theory). His resulting NS–α equations preserve Kelvin's theorem (1.4*b*) in the sense of the linkage mentioned, i.e. the HEL advective velocity $\boldsymbol{u}$ is replaced by the Eulerian mean velocity $\overline{\boldsymbol{v}^*}$, and $\overline{\boldsymbol{V}}$ is replaced by his second mean velocity. Unfortunately, as was shown by Soward & Roberts (2008), Holm's derivation was not carried out consistently to second order in $\boldsymbol{\xi}$ and, when corrected, it was found that Kelvin's theorem no longer held. An alternative treatment starting from the HEL formulation was accomplished by Roberts & Soward (2009).

The main aim of this paper is to construct a glm theory but from a different point of view to Holm. Whereas Holm starts from the Eulerian equations but makes use of the HEL-motivated representation (1.5) of the velocity, we take the HEL equations, which describe the state of the fluid at P^L as our starting point (see § 2) and seek an Eulerian representation of them (see § 3) in terms of state variables at P. Moffatt

(1986) introduced the essential idea needed to accomplish that objective. He realized the mapping (1.2) by use of a fictitious velocity field $\boldsymbol{\eta}^*$ that drags the point P to P^L. This is a concept that lies at the heart of differential geometry, where it is known as 'Lie dragging' (see e.g. Schutz 1980). Although Moffatt did not invoke Lie dragging, his procedure is tantamount to that process. The essential idea is that a tensor Lie dragged from P to P^L is still a tensor (i.e. in the sense that the coordinates x_i of P are regarded as non-Cartesian coordinates of P^L), exactly as required by our HEL formulation. Indeed, the increment of a Lie-dragged tensor over an infinitesimal displacement determines its Lie derivative, which is a tensor of the same type, e.g. $\mathrm{L}_{\boldsymbol{u}}\boldsymbol{V}$ in (1.3a) is a covariant vector like $\boldsymbol{V}$ and, for that matter, $\partial_t \boldsymbol{V}$ and $\nabla \Pi$.

One of the merits of Moffatt's approach is that we can regard the fictitious velocity field $\boldsymbol{\eta}^*$ as the prescribed vector field rather than the finite displacement $\boldsymbol{\xi} = \boldsymbol{x}^L - \boldsymbol{x}$ of P^L from P. The advantage is that $\boldsymbol{\xi}$, so obtained, exists and is unique. It is less obvious that the inverse problem, whereby $\boldsymbol{\xi}$ is prescribed and $\boldsymbol{\eta}^*$ is to be determined, has a solution, let alone whether it is unique. Although our small-amplitude expansions do, however, suggest existence and uniqueness of the solution for sufficiently small $\boldsymbol{\xi}$, McIntyre (1980) pointed out that there are possible configurations where the map $P \mapsto P^L$ might not be invertible, i.e. more than one P maps to P^L. With $\boldsymbol{\eta}^*$ prescribed, this complication cannot happen and the map $P \mapsto P^L$ is, by construction, invertible.

Our objective is to relate $\boldsymbol{V}(\boldsymbol{x}, t)$ and $\boldsymbol{u}(\boldsymbol{x}, t)$, which define the nature of the velocity at P^L, to the actual velocity $\boldsymbol{v}^*(\boldsymbol{x}, t)$ at P. Since they define conditions at different points P^L and P, we, like Moffatt, need to invoke Taylor series expansions generated by the finite displacement $\boldsymbol{\xi} = \boldsymbol{x}^L - \boldsymbol{x}$ of P^L from P. Indeed, we show in § 3 that the appropriate derivative in the expansions is the Lie derivative, L_{η}, based on the fictitious (or dragging) velocity field $\boldsymbol{\eta}^*$ rather than the more obvious expansion parameter $\boldsymbol{\xi}$. In this way we develop expansions to all orders of HEL variables at position P^L in terms of their Eulerian counterparts at position P, rather than P^L. Indeed, for small displacements, Moffatt chose $\boldsymbol{\eta}^*$ to be close to $\boldsymbol{\xi}$, specifically $\boldsymbol{\eta}^* = \boldsymbol{\xi} + O(\xi^2)$. Despite their closeness, we stress that the change of variables from the displacement $\boldsymbol{\xi}$ to the dragging velocity $\boldsymbol{\eta}^*$ is not cosmetic. Rather $\boldsymbol{\eta}^*$ is the natural and more useful variable to adopt for this approach, which we refer to as Euler transformed HEL (ETHEL), when important effects at $O(\xi^2)$ and higher are considered. Since the HEL equations are retained, the mean Kelvin's circulation theorem (1.4b) continues to hold for ETHEL.

The GLM $\rightarrow$ glm transformation above was motivated by the particular example of the $\boldsymbol{x} \mapsto \boldsymbol{x}^L$ mapping that arises in the statistical representation of a stochastically varying flow, but the basic analytical problem of representing an HEL description in Eulerian terms is independent of that example. Likewise, our Lie-dragged formulation of HEL in § 3, which is the heart of this paper, is also independent of the GLM application and so we prefer the term ETHEL over glm. In § 4 we return to the statistical application and briefly consider its implications with respect to the NS–α theory. Indeed, to glean the main thrust of our study, the technical development of §§ 3 and 4 could be skipped on first reading, i.e. after digesting § 2, which summarizes well-known results and develops notation, the reader might benefit from moving directly to § 5, where our main findings are discussed briefly. We also attempt to help the reader by placing technical matters, upon which the main direction of the arguments build, into appendices.

Before we continue, we add some cautionary words about notation. We find that neither the notation we employed in Roberts & Soward (2006a), which was guided by the notation of general tensor calculus, nor the alternative notation of Soward &

Roberts (2008), which follows Andrews & McIntyre (1978*a*) more closely, is adequate to formulate the ETHEL procedure clearly. In fact, we use a hybrid version of the two notations, which is more pedantic but less ambiguous. In making any direct comparisons with those papers, one must guard against overlooking subtle differences in notation. We do not apologise for this evolution, which is common in the differential geometry literature, where it is driven by more sophisticated usage (see e.g. Schutz 1980). Indeed, it is unfortunate that the symbol L_{η} cannot be used throughout to denote the Lie derivative, as it would obviate the need to use the distinct notations $\{\boldsymbol{\eta},\ \}$ and $[\boldsymbol{\eta},\ ]$ (see Appendix B) for the Lie derivative of covariant and contravariant vectors. The reason is that our ETHEL development of § 3 relies on the evaluation of the Lie derivatives of vectors at the undisplaced position P, where, at this truly Cartesian location, their covariant or contravariant nature is indistinguishable.

2. The hybrid Euler–Lagrange formulation

The Lagrangian representation of a fluid flow is thought of as a mapping, $\boldsymbol{x} \mapsto \boldsymbol{x}^{\Lambda}(\boldsymbol{x}, t)$, that takes a material particle initially located at $\boldsymbol{x}$ to its current position $\boldsymbol{x}^{\Lambda}$. Such a mapping is not optimal when dealing with stochastically varying flows, for which a preferable description may be the alternative mapping

$$\boldsymbol{x} \mapsto \boldsymbol{x}^{L}(\boldsymbol{x}, t) \equiv \boldsymbol{x} + \boldsymbol{\xi}(\boldsymbol{x}, t), \tag{2.1}$$

where $\boldsymbol{\xi}$ is the displacement of a fluid particle at $P^{L} : \boldsymbol{x}^{L}$ away from some reference position $P : \boldsymbol{x}$ it would occupy if it followed the mean flow. Evidently, this is neither an Eulerian nor a Lagrangian representation, but is a mixture of the two. This is why it is called an HEL description. Application of the HEL technique to mean flows may be found in Andrews & McIntyre (1978*a,b*), Holm (2002), Soward & Roberts (2008), Roberts & Soward (2009) and later in § 4. We follow the HEL development of Andrews & McIntyre (1978*a*) exactly but express the HEL variables in their ETHEL form (see § 3). Essential differences, however, emerge when averages are taken. The GLM objective, outlined in § 4.1, is achieved under the assumption that the mean displacement vanishes ($\overline{\boldsymbol{\xi}} = \boldsymbol{0}$, see (4.5*a*)), whereas our glm equations, developed in § 4.2, build on the alternative mean assumption $\overline{\boldsymbol{\eta}} = \boldsymbol{0}$ (see (4.8)), which we find more appropriate to the ETHEL formulation.

Evaluating a function at P^{L}: $\boldsymbol{x}^{L} \equiv (x_1^{L}, x_2^{L}, x_3^{L})$ rather than the reference position P: $\boldsymbol{x} \equiv (x_1, x_2, x_3)$ may be thought of as applying an HEL operator ^{L}. Our usage of it is motivated by the $^{\xi}$ operator of Andrews & McIntyre (1978*a*, equations (2.2)–(2.6)). So the values of the velocity $\boldsymbol{v}^{*}$ and modified pressure Π^{*} at the HEL position P^{L}, defined by the mapping (2.1), are written as $(\boldsymbol{v}^{*})^{L}$ and $(\Pi^{*})^{L}$ or simply $\boldsymbol{v}^{*L}$ and Π^{*L}. These are regarded as functions of the HEL coordinates x_i rather than the HEL position vector $\boldsymbol{x}^{L}$:

$$\boldsymbol{v}^{*L}(\boldsymbol{x}, t) \equiv \boldsymbol{v}^{*}(\boldsymbol{x}^{L}(\boldsymbol{x}, t), t), \qquad \Pi^{*L}(\boldsymbol{x}, t) \equiv \Pi^{*}(\boldsymbol{x}^{L}(\boldsymbol{x}, t), t). \tag{2.2a,b}$$

The HEL transformation of vectors such as $\boldsymbol{v}^{*}$ under the mapping (2.1) is complicated by the fact that they are 'strained' and 'rotated' by the mapping. Specifically, their HEL counterparts are obtained from general tensor calculus and come in two forms, covariant and contravariant:

$$\boldsymbol{V}(\boldsymbol{x}, t) \equiv (\nabla \boldsymbol{x}^{L}) \cdot \boldsymbol{v}^{*L}(\boldsymbol{x}, t), \qquad \boldsymbol{v}(\boldsymbol{x}, t) \equiv \boldsymbol{v}^{*L}(\boldsymbol{x}, t) \cdot (\nabla \boldsymbol{x}^{L})^{-1}, \tag{2.3a,b}$$

where, in component form relative to Cartesian coordinates, we have introduced the notation $((\nabla \boldsymbol{x}^{L}) \cdot \boldsymbol{v}^{*L})_i = (\nabla \boldsymbol{x}^{L})_{ij} v_j^{*L}$ and $(\boldsymbol{v}^{*L} \cdot (\nabla \boldsymbol{x}^{L})^{-1})_i = v_j^{*L}((\nabla \boldsymbol{x}^{L})^{-1})_{ji}$. Here the components of the deformation matrix $\nabla \boldsymbol{x}^L$ and its inverse $(\nabla \boldsymbol{x}^L)^{-1}$ are

$$(\nabla \boldsymbol{x}^L)_{ij} \equiv \nabla_i x_j^L, \quad \left((\nabla \boldsymbol{x}^L)^{-1}\right)_{ji} \equiv \nabla_j^L x_i, \tag{2.3c,d}$$

where $\nabla_i \equiv \partial/\partial x_i$ and $\nabla_j^L \equiv \partial/\partial x_j^L$. Here, as elsewhere, we use upper and lower case bold face letters for covariant and contravariant vectors, respectively.

Consider any scalar function $\psi^*(\boldsymbol{x}, t)$, from which we define $\psi^{*L}(\boldsymbol{x}, t) \equiv \psi^*(\boldsymbol{x}^L(\boldsymbol{x}, t), t)$ as in (2.2*b*). On application of the chain rule for differentiation to $\psi^*(\boldsymbol{x}^L(\boldsymbol{x}, t), t)$ and noting that $\nabla^L(\psi^*(\boldsymbol{x}^L, t)) = (\nabla\psi^*)^L(\boldsymbol{x}, t)$, we obtain

$$\nabla \psi^{*L} = (\nabla \boldsymbol{x}^L) \cdot (\nabla \psi^*)^L, \tag{2.4a}$$

implying that $(\nabla\psi^*)^L = (\nabla \boldsymbol{x}^L)^{-1} \cdot \nabla \psi^{*L}$. So on forming the scalar product $\boldsymbol{v}^{*L} \cdot (\nabla\psi^*)^L$, in which $\boldsymbol{v}^{*L} = \boldsymbol{v} \cdot \nabla \boldsymbol{x}^L$ (see (2.3*b*)), we arrive at the useful identity

$$(\boldsymbol{v}^* \cdot \nabla \psi^*)^L = (\boldsymbol{v} \cdot \nabla) \psi^{*L}, \tag{2.4b}$$

which is equivalent to (2.12*b*) below. When ψ^{*L} is a scalar HEL variable such as modified pressure Π^{*L}, it is convenient to simply remove ^{*L}, and write

$$\Pi(\boldsymbol{x}, t) \equiv \Pi^{*L}(\boldsymbol{x}, t). \tag{2.5}$$

A word of caution is needed here. Unlike the scalar Π^{*L} defined by (2.2*b*), the velocity vector $\boldsymbol{v}^{*L}$ defined by (2.2*a*) is definitely not an HEL variable. Only vectors such as $\boldsymbol{V}$ and $\boldsymbol{v}$ defined by (2.3*a*,*b*), which obey the transformation laws of the general tensor calculus, are HEL variables and we reserve the omission of the ^{*L} to them. Further useful results involving the gradients of HEL vectors are given in (B 2) of Appendix B.

A complication of the HEL map (2.1) is its time dependence. In fact, following our notation for the velocity $\boldsymbol{v}^*$, we write

$$\boldsymbol{w}^{*L}(\boldsymbol{x}, t) \equiv \partial_t \boldsymbol{x}^L(\boldsymbol{x}, t) \ (= \partial_t \boldsymbol{\xi}), \tag{2.6a}$$

for the velocity of the HEL position P^L. From (2.6*a*) we define the velocity field $\boldsymbol{w}^*(\boldsymbol{x}, t)$ at the reference position P implicitly by

$$\boldsymbol{w}^*(\boldsymbol{x}^L(\boldsymbol{x}, t), t) \equiv \boldsymbol{w}^{*L}(\boldsymbol{x}, t) \equiv \boldsymbol{w}(\boldsymbol{x}, t) \cdot \nabla \boldsymbol{x}^L, \tag{2.6b}$$

where $\boldsymbol{w}(\boldsymbol{x}, t)$ is its contravariant HEL form; *à propos* of our remarks above we emphasize that $\boldsymbol{w}^{*L}$ is not an HEL vector. Equations (2.6*a*,*b*) together imply that

$$\partial_t \boldsymbol{x}^L = \boldsymbol{w} \cdot \nabla \boldsymbol{x}^L. \tag{2.6c}$$

We may associate with the flow velocity $\boldsymbol{w}^*$ (see (2.6*b*)) a material derivative $\partial_t + \boldsymbol{w}^* \cdot \nabla$. When this acts on the arbitrary scalar function ψ^*, introduced above, and is evaluated at the point $\boldsymbol{x}^L$ moving with that flow, we write $(\partial_t\psi^* + \boldsymbol{w}^* \cdot \nabla\psi^*)^L$, which is simply the time derivative of $\psi^{*L}(\boldsymbol{x}, t) \equiv \psi^*(\boldsymbol{x}^L(\boldsymbol{x}, t), t)$, namely $\partial_t \psi^{*L}$ $(\equiv \partial_t(\psi^{*L}))$. On writing $(\boldsymbol{w}^* \cdot \nabla\psi^*)^L = (\boldsymbol{w} \cdot \nabla)\psi^{*L}$ (see (2.4*b*)), the relation between the time derivatives becomes $(\partial_t\psi^*)^L + (\boldsymbol{w} \cdot \nabla)\psi^{*L} = \partial_t\psi^{*L}$, which when rearranged yields the fundamental relation

$$(\partial_t \psi^*)^L = (\partial_t - \boldsymbol{w} \cdot \nabla)\, \psi^{*L}. \tag{2.7}$$

Substitution of $\psi^* = x_i$ into (2.7) recovers the result $(\partial_t - \boldsymbol{w} \cdot \nabla)\boldsymbol{x}^L = \boldsymbol{0}$ (see (2.6*c*)).

The material derivative of ψ^* at the HEL position $\boldsymbol{x}^L$ is determined from (1.1*c*), (2.4*b*) and (2.7) as

$$(\mathscr{D}_t \psi^*)^L = \mathrm{D}_t(\psi^{*L}) \equiv \mathrm{D}_t \psi^{*L}, \tag{2.8a}$$

where

$$\mathrm{D}_t \equiv \partial_t + \boldsymbol{u}(\boldsymbol{x},t)\cdot\nabla \quad \text{and} \quad \boldsymbol{u} \equiv \boldsymbol{v} - \boldsymbol{w}. \tag{2.8b,c}$$

If we define the Eulerian vector field $\boldsymbol{u}^*(\boldsymbol{x},t)$ implicitly by

$$\boldsymbol{u}^*(\boldsymbol{x}^L(\boldsymbol{x},t),t) \equiv \boldsymbol{u}^{*L}(\boldsymbol{x},t) \equiv \boldsymbol{u}(\boldsymbol{x},t)\cdot\nabla\boldsymbol{x}^L\,(=\boldsymbol{u}+\boldsymbol{u}\cdot\nabla\boldsymbol{\xi}), \tag{2.9a}$$

we consistently have, for the choice $\psi^* = x_i$ in (2.8a) and use of (2.6a), that

$$\boldsymbol{v}^{*L} = (\mathscr{D}_t\boldsymbol{x})^L = \mathrm{D}_t\boldsymbol{x}^L = \boldsymbol{u} + \mathrm{D}_t\boldsymbol{\xi} = \boldsymbol{u}^{*L} + \boldsymbol{w}^{*L}. \tag{2.9b}$$

This exposes $\boldsymbol{u}^{*L}$ as the advective velocity at $\boldsymbol{x}^L$ with the velocity $\boldsymbol{w}^{*L}$ of $\boldsymbol{x}^L$ relative to $\boldsymbol{x}$ removed from $\boldsymbol{v}^{*L}$. In the case of a pure Lagrangian description, for which $\boldsymbol{x}^L$ follows the fluid particle motion and $\boldsymbol{v}^{*L} = \boldsymbol{w}^{*L}$, (2.9$b$) implies that $\boldsymbol{u}^{*L} = \boldsymbol{0}$.

We now apply our results together with further relations established in Appendix B to Euler's equation (1.1d). Since $(\nabla\boldsymbol{x}^L)\cdot(\nabla\Pi^*)^L = \nabla\Pi$ (see (2.4a)), we pre-multiply (1.1d), evaluated at the HEL position $\boldsymbol{x}^L$, with $\nabla\boldsymbol{x}^L$. The consequence for the term $(\partial_t\boldsymbol{v}^* + \boldsymbol{v}^*\cdot\nabla\boldsymbol{v}^* + \nabla(|\boldsymbol{v}^*|^2/2))^L$ on the left-hand side of (1.1d) is determined from (B 6a) with $\boldsymbol{f}^* = \boldsymbol{v}^*$. Combining these two results we obtain the covariant HEL form (1.3a) of Euler's equation, namely

$$\partial_t\boldsymbol{V} + \{\boldsymbol{u},\ \boldsymbol{V}\} = -\nabla\Pi, \tag{2.10}$$

in which $\{\boldsymbol{u},\boldsymbol{V}\} \equiv \boldsymbol{u}\cdot\nabla\boldsymbol{V} + (\nabla\boldsymbol{u})\cdot\boldsymbol{V}$ (see Andrews & McIntyre 1978a, Soward & Roberts 2008 or Roberts & Soward 2009).

It must be stressed that all the variables appearing in (2.10) are functions of $\boldsymbol{x}$ and t but they define properties at $\boldsymbol{x}^L(\boldsymbol{x},t)$. A form of Kelvin's theorem follows from (2.10):

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\mathscr{C}} \boldsymbol{V}\cdot\mathrm{d}\boldsymbol{x} = 0, \tag{2.11}$$

where $\mathscr{C}(t)$ is composed of HEL coordinates $\boldsymbol{x}$, which move in the HEL frame with velocity $\boldsymbol{u}$, i.e. $\mathscr{C}(t) \mapsto \mathscr{C}^*(t)$ under the map $\boldsymbol{x} \mapsto \boldsymbol{x}^L$. Indeed, the comparison with (1.1f) is completed by noting that (2.3a) implies the following:

$$\boldsymbol{V}(\boldsymbol{x},t)\cdot\mathrm{d}\boldsymbol{x} = \boldsymbol{v}^*(\boldsymbol{x}^L,t)\cdot\mathrm{d}\boldsymbol{x}^L. \tag{2.12a}$$

This should be compared with the identity

$$\boldsymbol{v}(\boldsymbol{x},t)\cdot\nabla = \boldsymbol{v}^*(\boldsymbol{x}^L,t)\cdot\nabla^L, \tag{2.12b}$$

which describes the content of (2.4b).

Since our velocity field $\boldsymbol{v}^*$ is solenoidal (see (1.1b)), it is natural in our ETHEL application in § 3 to impose the same restriction on $\boldsymbol{w}^*$ with the consequence that $\boldsymbol{u}^* = \boldsymbol{v}^* - \boldsymbol{w}^*$ (see (2.9b)) is solenoidal too:

$$(\nabla\cdot\boldsymbol{v}^*)^L = 0, \quad (\nabla\cdot\boldsymbol{w}^*)^L = 0, \quad (\nabla\cdot\boldsymbol{u}^*)^L = 0. \tag{2.13a,b,c}$$

On the basis of these assumptions we explain briefly in Appendix C that the corresponding contravariant vectors $\boldsymbol{v}$, $\boldsymbol{w}$ and $\boldsymbol{u}$ are also solenoidal:

$$\nabla\cdot\boldsymbol{v} = 0, \quad \nabla\cdot\boldsymbol{w} = 0, \quad \nabla\cdot\boldsymbol{u} = 0. \tag{2.13d,e,f}$$

3. Eulerian transformed HEL formulation

Our objective in this section is to relate the HEL variables, such as $\boldsymbol{V}(\boldsymbol{x},t)$, $\boldsymbol{u}(\boldsymbol{x},t)$ and $\Pi(\boldsymbol{x},t)$, which describe the state of the fluid at the displaced HEL position P^L, to the Eulerian values of $\boldsymbol{v}^*(\boldsymbol{x},t)$ and $\Pi^*(\boldsymbol{x},t)$ at P itself rather than $\boldsymbol{v}^*(\boldsymbol{x}^L,t)$ and $\Pi^*(\boldsymbol{x}^L,t)$ at P^L. From this point of view, the HEL coordinates x_i are truly Eulerian coordinates and, via the relations that we construct, the vectors $\boldsymbol{V}(\boldsymbol{x},t)$, $\boldsymbol{u}(\boldsymbol{x},t)$ and scalar $\Pi(\boldsymbol{x},t)$, which have their roots in general tensor concepts, have a clear Eulerian interpretation. This is why we call our construction an ETHEL formulation. As explained in § 1, a partial appreciation of the ETHEL construction, which omits the intermediate HEL step, has been employed before; see e.g. Braginsky's use of 'effective variables' (Braginsky 1964*a*,*b*), Tough & Roberts (1968), who identified their importance in the equation of motion, and Soward (1971*a*). A key idea in what follows is contained in Soward (1971*b*), where power series for Braginsky's effective variables were given explicitly to all orders. Indeed, these results provided the motivation for the HEL development of Soward (1972). Here we will pursue that idea in the reverse ETHEL order.

Fortunately, the techniques that we use are well known in differential geometry (see e.g. Schutz 1980), but since fluid mechanicists are generally not familiar with these methods, we develop the ideas from scratch sufficient for our application. A relatively straightforward description of the tensor calculus and the Lie derivatives needed, which avoids unfamiliar technicalities, is given by d'Inverno (1992). We comment briefly on the relation of tensor calculus to our development in Appendix A.

3.1. *The HEL dragging velocity: contravariant* $\boldsymbol{\eta}$

Following Moffatt (1986), our strategy is to find an Eulerian recipe to effect the HEL map $\boldsymbol{x} \mapsto \boldsymbol{x}^L(\boldsymbol{x})$. Here to avoid confusion, we temporarily suppress reference to the dependence on t, which is irrelevant to our immediate considerations. We achieve our objective via the introduction of an additional mapping

$$\boldsymbol{x} \mapsto \boldsymbol{x}^\ell(\boldsymbol{x},\tau) \equiv \boldsymbol{x} + \boldsymbol{\xi}^\ell(\boldsymbol{x},\tau) \quad \text{on} \quad 0 \leqslant \tau \leqslant 1, \tag{3.1a}$$

which is infinitely differentiable in the parameter τ. The position $P^\ell : \boldsymbol{x}^\ell \equiv (x_1^\ell, x_2^\ell, x_3^\ell)$ has the properties

$$\boldsymbol{x}^\ell(\boldsymbol{x},0) = \boldsymbol{x} \quad \text{and} \quad \boldsymbol{x}^\ell(\boldsymbol{x},1) = \boldsymbol{x}^L(\boldsymbol{x}), \tag{3.1b,c}$$

so that as τ 'evolves' from 0 to 1, P^ℓ is dragged continuously from the reference position P at $\tau = 0$ to the HEL position P^L at $\tau = 1$. Along this path $\boldsymbol{x}^\ell(\boldsymbol{x},\tau)$ has the Taylor series expansion

$$\boldsymbol{x}^\ell(\boldsymbol{x},\tau) = \exp(\tau\partial_\tau)\,\boldsymbol{x}^\ell \equiv \boldsymbol{x}^\ell + \tau\partial_\tau\,\boldsymbol{x}^\ell + \tfrac{1}{2}\tau^2\partial_\tau^2\,\boldsymbol{x}^\ell + \cdots \tag{3.2a}$$

(see Schutz 1980, equation (2.6)), which, at the end point P^L, takes the specific value

$$\boldsymbol{x}^\ell(\boldsymbol{x},1) = \exp(\partial_\tau)\,\boldsymbol{x}^\ell, \tag{3.2b}$$

where $\boldsymbol{x}^\ell$ and its derivatives $\partial_\tau^n \boldsymbol{x}^\ell$ ($n \geqslant 1$) on the right-hand sides of (3.2*a*,*b*) are evaluated at $\tau = 0$.

On regarding τ as the fictitious time variable, the fictitious motion of P^ℓ (for fixed $\boldsymbol{x}$) determines its dragging velocity

$$\boldsymbol{\eta}^{*\ell}(\boldsymbol{x},\tau) \equiv \partial_\tau\,\boldsymbol{x}^\ell, \tag{3.3a}$$

at $\boldsymbol{x}^\ell$ (cf. (2.6*a*)). Our construction of the τ-map $\boldsymbol{x} \mapsto \boldsymbol{x}^\ell(\boldsymbol{x},\tau)$ is identical in mathematical structure to the map $\boldsymbol{x} \mapsto \boldsymbol{x}^L(\boldsymbol{x},t)$. Consequently, results of § 2 may be taken over intact by the re-labelling $t \to \tau$, $\boldsymbol{w} \to \boldsymbol{\eta}$ and ${}^L \to {}^\ell$. Accordingly, from

(3.3a) we define the dragging velocity $\boldsymbol{\eta}^*(\boldsymbol{x},\tau)$ at position $\boldsymbol{x}$ implicitly by

$$\boldsymbol{\eta}^*(\boldsymbol{x}^\ell(\boldsymbol{x},\tau),\tau) \equiv \boldsymbol{\eta}^{*\ell}(\boldsymbol{x},\tau) \equiv \boldsymbol{\eta}^\ell(\boldsymbol{x},\tau)\cdot\nabla\boldsymbol{x}^\ell(\boldsymbol{x},\tau), \tag{3.3b}$$

where $\boldsymbol{\eta}^\ell$ is the corresponding contravariant dragging velocity (cf. (2.6b), but note our additional superscript $^\ell$, which we include for notational consistency with $\boldsymbol{v}^\ell$ defined by (3.17b) below). With this definition of $\boldsymbol{\eta}^\ell$, we may express (3.3a) in the alternative form

$$\partial_\tau \boldsymbol{x}^\ell = \boldsymbol{\eta}^\ell\cdot\nabla\boldsymbol{x}^\ell \tag{3.3c}$$

(cf. (2.6c)). When (3.3b) is evaluated at the initial point $\boldsymbol{x}^\ell(\boldsymbol{x},0)=\boldsymbol{x}$ and final point $\boldsymbol{x}^\ell(\boldsymbol{x},1)=\boldsymbol{x}^L(\boldsymbol{x})$ (see (3.1b,c)), we obtain

$$\boldsymbol{\eta}^{*\ell}(\boldsymbol{x},0) = \boldsymbol{\eta}^\ell(\boldsymbol{x},0), \quad \boldsymbol{\eta}^{*\ell}(\boldsymbol{x},1) = \boldsymbol{\eta}(\boldsymbol{x})\cdot\nabla\boldsymbol{x}^L(\boldsymbol{x}), \tag{3.4a,b}$$

where

$$\boldsymbol{\eta}(\boldsymbol{x}) \equiv \boldsymbol{\eta}^\ell(\boldsymbol{x},1). \tag{3.4c}$$

To evaluate the terms on the right-hand sides of (3.2a,b), we need to determine the τ-derivatives of (3.3b). To simplify matters we assume that $\boldsymbol{\eta}^\ell$ is τ-steady so that $\partial_\tau\boldsymbol{\eta}^\ell=\boldsymbol{0}$. Then on making the substitution $\boldsymbol{\eta}^\ell=\boldsymbol{\eta}$, independent of τ, into (3.3c), the first derivative of $\partial_\tau \boldsymbol{x}^\ell$ is simply

$$\partial_\tau^2 \boldsymbol{x}^\ell = (\boldsymbol{\eta}\cdot\nabla)\,\partial_\tau \boldsymbol{x}^\ell = (\boldsymbol{\eta}\cdot\nabla)^2\boldsymbol{x}^\ell. \tag{3.5a}$$

Likewise, repeated differentiation gives the general result

$$\partial_\tau^n \boldsymbol{x}^\ell = (\boldsymbol{\eta}\cdot\nabla)^n\boldsymbol{x}^\ell \quad \text{for} \quad n \geqslant 1. \tag{3.5b}$$

On substitution into (3.2a) and use of $\boldsymbol{x}^\ell(\boldsymbol{x},0)=\boldsymbol{x}$ we obtain

$$\boldsymbol{x}^\ell = \boldsymbol{x} + \tau(\boldsymbol{\eta}\cdot\nabla)\boldsymbol{x} + \tfrac{1}{2}\tau^2(\boldsymbol{\eta}\cdot\nabla)^2\boldsymbol{x} + \cdots, \tag{3.6a}$$

which determines

$$\boldsymbol{\xi}^\ell = \tau\boldsymbol{\eta} + \tfrac{1}{2}\tau^2\boldsymbol{\eta}\cdot\nabla\boldsymbol{\eta} + \frac{1}{3!}\tau^3(\boldsymbol{\eta}\cdot\nabla)^2\boldsymbol{\eta} + \cdots. \tag{3.6b}$$

From (3.6a) the value of $\boldsymbol{x}^\ell$ at $\tau=1$ (see (3.2b)) is

$$\boldsymbol{x}^L = \boldsymbol{x} + (\boldsymbol{\eta}\cdot\nabla)\boldsymbol{x} + \tfrac{1}{2}(\boldsymbol{\eta}\cdot\nabla)^2\boldsymbol{x} + \cdots \tag{3.6c}$$

$$\equiv \exp(\boldsymbol{\eta}\cdot\nabla)\,\boldsymbol{x}. \tag{3.6d}$$

Finally, the HEL displacement $\boldsymbol{\xi}(\boldsymbol{x})=\boldsymbol{\xi}^\ell(\boldsymbol{x},1)$ is determined by (3.6b) as

$$\boldsymbol{\xi} = \boldsymbol{\eta} + \tfrac{1}{2}\boldsymbol{\eta}\cdot\nabla\boldsymbol{\eta} + \frac{1}{3!}(\boldsymbol{\eta}\cdot\nabla)^2\boldsymbol{\eta} + \cdots. \tag{3.7}$$

Our development, though similar to Moffatt (1986), has a different starting point. Whereas we have built on the τ-steadiness of $\boldsymbol{\eta}$, which by (3.4b) is an HEL variable, Moffatt assumes that the fictitious flow $\boldsymbol{\eta}^*$ (an Eulerian variable) is τ-steady. Curiously, they are the same (see (3.11c)) as we now show. To establish their equivalence, we note that the relation (B 5a) (remarkable in view of (B 5b)) translates to

$$(\partial_\tau\boldsymbol{\eta}^*)^\ell = (\partial_\tau\boldsymbol{\eta}^\ell)\cdot(\nabla\boldsymbol{x}^\ell). \tag{3.8}$$

An important consequence is that if either $\boldsymbol{\eta}^*$ or $\boldsymbol{\eta}^\ell$ is τ-steady, so is the other:

$$\partial_\tau\boldsymbol{\eta}^* = \boldsymbol{0} \quad \Longleftrightarrow \quad \partial_\tau\boldsymbol{\eta}^\ell = \boldsymbol{0}. \tag{3.9a,b}$$

In order to dispel a possible point of confusion, we note that the case of τ-steady $\boldsymbol{\eta}^{*\ell}$, for which

$$\boldsymbol{\xi}^{\ell}(\boldsymbol{x},\tau)=\tau\,\boldsymbol{\xi}(\boldsymbol{x}),\quad \boldsymbol{\eta}^{*\ell}(\boldsymbol{x})=\boldsymbol{\xi}(\boldsymbol{x}) \tag{3.10a,b}$$

(see (3.3*a*)), does not correspond to τ-steady $\boldsymbol{\eta}^{\ell}$ because (3.3*b*) has the unsteady solution

$$\boldsymbol{\eta}^{\ell}(\boldsymbol{x},\tau)=\boldsymbol{\xi}\cdot(\boldsymbol{I}+\tau\,\nabla\boldsymbol{\xi})^{-1}. \tag{3.10c}$$

Since we have already assumed that $\boldsymbol{\eta}^{\ell}$ is τ-steady, it follows from (3.9*a*) that $\boldsymbol{\eta}^{*}$ is τ-steady too. From (3.3*b*) and (3.4*c*), it immediately follows that

$$\boldsymbol{\eta}^{*\ell}(\boldsymbol{x},\tau)=\boldsymbol{\eta}(\boldsymbol{x})\cdot\nabla\boldsymbol{x}^{\ell}(\boldsymbol{x},\tau)\quad\text{with}\quad \boldsymbol{\eta}^{\ell}(\boldsymbol{x},\tau)=\boldsymbol{\eta}(\boldsymbol{x}). \tag{3.11a,b}$$

More specifically, (3.11*a*) evaluated at $\tau=0$ and 1 (see (3.4*a,b*) and (3.1*b,c*) again), respectively, yields

$$\boldsymbol{\eta}^{*}(\boldsymbol{x})=\boldsymbol{\eta}(\boldsymbol{x}),\quad \boldsymbol{\eta}^{*L}(\boldsymbol{x})=\boldsymbol{\eta}(\boldsymbol{x})\cdot\nabla\boldsymbol{x}^{L}(\boldsymbol{x}), \tag{3.11c,d}$$

where $\boldsymbol{\eta}^{*L}(\boldsymbol{x})\equiv\boldsymbol{\eta}^{*}(\boldsymbol{x}^{L}(\boldsymbol{x}))=\boldsymbol{\eta}^{*\ell}(\boldsymbol{x},1)$. The latter (i.e. (3.11*d*)) shows, via (B 1*b*), that $\boldsymbol{\eta}$ is the contravariant HEL variable version of the dragging velocity $\boldsymbol{\eta}^{*}$. Together with the former (i.e. (3.11*c*)), we obtain the non-trivial relation $\boldsymbol{\eta}^{*}(\boldsymbol{x}^{L}(\boldsymbol{x}))=\boldsymbol{\eta}^{*}(\boldsymbol{x})\cdot\nabla\boldsymbol{x}^{L}(\boldsymbol{x})$ between $\boldsymbol{\eta}^{*}$ evaluated at two distinct points P^{L} and P (see also the explicit formula (5.2*a*) below). Interestingly, Davidson (2000) used the term 'virtual displacement' to describe $\boldsymbol{\eta}^{*}$ and attributes it, erroneously we believe, to Moffatt (1986).

Just as we assumed that the HEL map $\boldsymbol{x}\mapsto\boldsymbol{x}^{L}$ is isochoric (see Appendix C), we follow Moffatt and assume that the τ-map $\boldsymbol{x}\mapsto\boldsymbol{x}^{\ell}$ is isochoric too. As a consequence, we have, as in (2.13*e*), that

$$\nabla\cdot\boldsymbol{\eta}=0, \tag{3.12a}$$

whereas, in contrast with this equation, (3.7) implies that generally $\nabla\cdot\boldsymbol{\xi}\neq0$. In fact, writing (3.7) in the alternative form $\boldsymbol{\xi}=\boldsymbol{\eta}+(1/2)\nabla\cdot(\boldsymbol{\eta}\boldsymbol{\eta})+\cdots$, we see that

$$\nabla\cdot\boldsymbol{\xi}=\tfrac{1}{2}(\nabla\nabla):(\boldsymbol{\eta}\boldsymbol{\eta})+\cdots \tag{3.12b}$$

(in dyadic notation $(\nabla\nabla):(\boldsymbol{\xi}\boldsymbol{\xi})\equiv(\nabla_i\nabla_j)(\xi_i\xi_j)$).

In the subsections that follow, we express all HEL variables as functions of $\boldsymbol{\eta}$ rather than $\boldsymbol{\xi}$. Although this may be achieved by direct substitution of (3.6*c,d*), the simple structure of these variables is more readily reached by following the techniques developed in this subsection. To this end it is important to appreciate that all Lie-dragged variables of interest $\psi^{*}(\boldsymbol{x})$ (say) are independent of τ (i.e. $\partial_\tau\psi^{*}=0$ and see Schutz 1980, p. 74). So on translating the result (2.7) into our τ-notation, we have $(\partial_\tau-\boldsymbol{\eta}\cdot\nabla)\psi^{*\ell}=(\partial_\tau\psi^{*})^{\ell}=0$, whence

$$\partial_\tau\,\psi^{*\ell}=\boldsymbol{\eta}\cdot\nabla\,\psi^{*\ell}, \tag{3.13a}$$

where $\psi^{*\ell}(\boldsymbol{x},\tau)=\psi^{*}(\boldsymbol{x}^{\ell}(\boldsymbol{x},\tau))$ is the value of ψ^{*} at the dragged position $P^{\ell}:\boldsymbol{x}^{\ell}(\boldsymbol{x},\tau)$. The value of $\partial_\tau\psi^{*\ell}$, so obtained, defines the Lie derivative of the scalar ψ^{*} at P^{ℓ}:

$$\mathrm{L}_{\eta}\,\psi^{*\ell}\equiv\boldsymbol{\eta}\cdot\nabla\,\psi^{*\ell}. \tag{3.13b}$$

The same ideas apply to the Lie dragging of vectors and higher order tensors, with the consequence, as we will see, that their Lie derivatives are different.

3.2. *The HEL modified pressure: $\varPi$*

We apply Lie dragging to the scalar modified pressure $\varPi^*(\boldsymbol{x})$, which, according to (3.13), satisfies

$$\mathrm{L}_{\eta}\varPi^{*\ell} = \partial_\tau \varPi^{*\ell} = \boldsymbol{\eta}\cdot\nabla\varPi^{*\ell}. \tag{3.14a}$$

Then, since $\boldsymbol{\eta}(\boldsymbol{x})$ is independent of τ, repeated differentiation gives

$$\mathrm{L}_{\eta}^n\varPi^{*\ell} = \partial_\tau^n \varPi^{*\ell} = (\boldsymbol{\eta}\cdot\nabla)^n\varPi^{*\ell} \quad \text{for} \quad n \geqslant 1, \tag{3.14b}$$

similar to (3.5*b*). Armed with these results, we may now relate the HEL modified pressure $\varPi(\boldsymbol{x}) \equiv \varPi^{*L}(\boldsymbol{x}) = \varPi^{*\ell}(\boldsymbol{x}, 1)$ to the Eulerian modified pressure $\varPi^*(\boldsymbol{x}) = \varPi^{*\ell}(\boldsymbol{x}, 0)$ by the Taylor series expansion

$$\varPi^{*\ell}(\boldsymbol{x}, 1) = \left(\varPi^{*\ell} + \mathrm{L}_{\eta}\varPi^{*\ell} + \tfrac{1}{2}\mathrm{L}_{\eta}^2\varPi^{*\ell} + \cdots\right)(\boldsymbol{x}, 0). \tag{3.15}$$

It follows from (3.15) that the modified pressure $\varPi^*$, evaluated at the HEL position $P^L : \boldsymbol{x}^L(\boldsymbol{x})$ (namely the HEL value $\varPi(\boldsymbol{x})$), is related to its value $\varPi^*(\boldsymbol{x})$ at the reference position $P : \boldsymbol{x}$ by

$$\varPi(\boldsymbol{x}) = \varPi^* + \boldsymbol{\eta}\cdot\nabla\varPi^* + \tfrac{1}{2}(\boldsymbol{\eta}\cdot\nabla)^2\varPi^* + \cdots \tag{3.16a}$$

$$= \exp(\boldsymbol{\eta}\cdot\nabla)\varPi^*(\boldsymbol{x}). \tag{3.16b}$$

The result (3.16*b*) may also be inverted:

$$\varPi^* = \exp(-\mathrm{L}_{\eta})\varPi, \quad \mathrm{L}_{\eta}\varPi \equiv \boldsymbol{\eta}\cdot\nabla\varPi, \tag{3.16c}$$

where $\mathrm{L}_{\eta}\varPi$ is the Lie derivative of scalars defined by (3.13*b*) (see d'Inverno 1992, equation (6.14)). The Taylor series, which (3.16*c*) defines, is similar to (3.16*a*) but with the interchanges $\varPi^* \leftrightarrow \varPi$ and $\boldsymbol{\eta} \leftrightarrow -\boldsymbol{\eta}$.

3.3. *The HEL velocities: covariant $\boldsymbol{V}$ and contravariant $\boldsymbol{v}$*

The covariant and contravariant forms of the velocity $\boldsymbol{v}^*(\boldsymbol{x})$ at the dragged position $P^\ell : \boldsymbol{x}^\ell(\boldsymbol{x}, \tau)$ are

$$\boldsymbol{V}^\ell \equiv (\nabla\boldsymbol{x}^\ell)\cdot\boldsymbol{v}^{*\ell} \quad \text{and} \quad \boldsymbol{v}^\ell \equiv \boldsymbol{v}^{*\ell}\cdot(\nabla\boldsymbol{x}^\ell)^{-1}, \tag{3.17a,b}$$

respectively. At $\tau = 0$, where $\boldsymbol{x}^\ell = \boldsymbol{x}$, they simply take the Eulerian value

$$\boldsymbol{V}^\ell(\boldsymbol{x}, 0) = \boldsymbol{v}^\ell(\boldsymbol{x}, 0) = \boldsymbol{v}^*(\boldsymbol{x}). \tag{3.18}$$

To avoid confusion, it is necessary to include the $^\ell$ superscripts on the left-hand sides of (3.17*a,b*) because the idea expressed below (3.3*a*), of translating the t-maps into corresponding results for τ-maps by changing ^{L} to $^\ell$, fails for (3.17*a,b*), since $\boldsymbol{V}^\ell$ and $\boldsymbol{v}^\ell$ defined in (3.17*a,b*) coincide with the HEL variables $\boldsymbol{V}$ and $\boldsymbol{v}$ only for $\tau = 1$:

$$\boldsymbol{V}(\boldsymbol{x}) = (\nabla\boldsymbol{x}^L)\cdot\boldsymbol{v}^{*L} = \boldsymbol{V}^\ell(\boldsymbol{x}, 1), \tag{3.19a}$$

$$\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{v}^{*L}\cdot(\nabla\boldsymbol{x}^L)^{-1} = \boldsymbol{v}^\ell(\boldsymbol{x}, 1). \tag{3.19b}$$

Since $\boldsymbol{v}^*(\boldsymbol{x})$ is τ-steady, we have the Lie dragging property $(\partial_\tau\boldsymbol{v}^*)^\ell = \boldsymbol{0}$. So use of (B 4*a,b*), expressed in τ-time notation with $\boldsymbol{f}^* = \boldsymbol{v}^*$, determines the Lie derivatives

$$\mathrm{L}_{\eta}\boldsymbol{V}^\ell \equiv \partial_\tau\boldsymbol{V}^\ell = \{\boldsymbol{\eta},\ \boldsymbol{V}^\ell\}, \quad \mathrm{L}_{\eta}\boldsymbol{v}^\ell \equiv \partial_\tau\boldsymbol{v}^\ell = [\boldsymbol{\eta},\ \boldsymbol{v}^\ell], \tag{3.20a,b}$$

appropriate to covariant and contravariant vector fields (see d'Inverno 1992, equations (6.16) and (6.15)), respectively, where the bilinear operators $\{\,,\,\}$ and $[\,,\,]$ are defined by (B 2*a,b*).

As in § 3.2, we employ the Taylor expansions

$$\boldsymbol{V}^{\ell}(\boldsymbol{x},1)=\exp(\mathrm{L}_{\eta})\,\boldsymbol{V}^{\ell}(\boldsymbol{x},0),\quad \boldsymbol{v}^{\ell}(\boldsymbol{x},1)=\exp(\mathrm{L}_{\eta})\,\boldsymbol{v}^{\ell}(\boldsymbol{x},0), \tag{3.21a,b}$$

to determine the values of $\boldsymbol{V}$ and $\boldsymbol{v}$ (see (3.19*a,b*)). Then repeated use of (3.20), noting (3.18), gives

$$\boldsymbol{V}(\boldsymbol{x})=\exp\{\boldsymbol{\eta},\,\bullet\}\boldsymbol{v}^{*}=\boldsymbol{v}^{*}+\{\boldsymbol{\eta},\,\boldsymbol{v}^{*}\}+\tfrac{1}{2}\{\boldsymbol{\eta},\,\{\boldsymbol{\eta},\,\boldsymbol{v}^{*}\}\}+\cdots, \tag{3.22a}$$

$$\boldsymbol{v}(\boldsymbol{x})=\exp[\boldsymbol{\eta},\,\bullet]\boldsymbol{v}^{*}=\boldsymbol{v}^{*}+[\boldsymbol{\eta},\,\boldsymbol{v}^{*}]+\tfrac{1}{2}[\boldsymbol{\eta},\,[\boldsymbol{\eta},\,\boldsymbol{v}^{*}]]+\cdots, \tag{3.22b}$$

where $\bullet$ is a marker to emphasize the location of the object upon which the operation that has been exponentiated acts. The covariant and contravariant vectors $\boldsymbol{V}$ and $\boldsymbol{v}$, which describe properties of the velocity $\boldsymbol{v}^{*L}$ at the HEL position $P^{L}:\boldsymbol{x}^{L}(\boldsymbol{x})$, are defined via (3.22*a,b*) in terms of the velocity $\boldsymbol{v}^{*}$ at the reference position $P:\boldsymbol{x}$. In this sense we say that (3.22*a,b*) provides an Eulerian description of the HEL system. Moreover, they have inverses

$$\boldsymbol{v}^{*}=\begin{cases}\exp(-\mathrm{L}_{\eta})\,\boldsymbol{V}, & \mathrm{L}_{\eta}\boldsymbol{V}\equiv\{\boldsymbol{\eta},\,\boldsymbol{V}\},\\ \exp(-\mathrm{L}_{\eta})\,\boldsymbol{v}, & \mathrm{L}_{\eta}\,\boldsymbol{v}\equiv[\,\boldsymbol{\eta},\,\boldsymbol{v}\,]\end{cases} \tag{3.22c}$$

(cf. (3.16*c*)), which are essentially (3.22*a,b*) after the interchange $\boldsymbol{\eta}\leftrightarrow-\boldsymbol{\eta}$ with $\boldsymbol{v}^{*}\leftrightarrow\boldsymbol{V}$ in (3.22*a*) and $\boldsymbol{v}^{*}\leftrightarrow\boldsymbol{v}$ in (3.22*b*).

Finally, since both $\boldsymbol{v}^{*}$ and $\boldsymbol{\eta}$ are solenoidal (see (1.1*b*) and (3.12*a*)) we may express $\boldsymbol{v}$ given by (3.22*b*) in the alternative form

$$\boldsymbol{v}=\boldsymbol{v}^{*}-\nabla\times(\boldsymbol{v}^{*}\times\boldsymbol{\eta})+\tfrac{1}{2}\nabla\times((\nabla\times(\boldsymbol{v}^{*}\times\boldsymbol{\eta}))\times\boldsymbol{\eta})-\cdots \tag{3.23a}$$

$$=\exp(\nabla\times(\boldsymbol{\eta}\times\bullet))\,\boldsymbol{v}^{*} \tag{3.23b}$$

(cf. Moffatt 1986, formulae (2.9)–(2.11) for magnetic field and his formulae (3.3)–(3.6) for vorticity, both of which are solenoidal vectors that have been transformed contravariantly like $\boldsymbol{v}^{*}$ here; Moffatt has no formulae for covariant vectors like $\boldsymbol{V}$). Since $\boldsymbol{v}$ is also solenoidal (see (2.13*d*)), we may express (3.22*c*) in the similar form

$$\boldsymbol{v}^{*}=\exp(-\nabla\times(\boldsymbol{\eta}\times\bullet))\boldsymbol{v}. \tag{3.23c}$$

3.4. *The advective contravariant HEL velocity* $\boldsymbol{u}$

Provided that no time derivatives are involved, the analysis of the previous subsections applies even when the HEL displacement $\boldsymbol{x}^{L}(\boldsymbol{x},t)$ and $\boldsymbol{\eta}(\boldsymbol{x},t)$ depend explicitly on t. Nevertheless, to determine the time derivative $\partial_t\boldsymbol{x}^{L}$, we need to consider the time-dependent fictitious flow $\boldsymbol{x}\mapsto\boldsymbol{x}^{\ell}(\boldsymbol{x},t,\tau)$, such that $\boldsymbol{\eta}^{*\ell}(\boldsymbol{x},t,\tau)$ defined by (3.3*a*) is explicitly dependent on t. Significantly, the dragging velocity $\boldsymbol{\eta}^{*}(\boldsymbol{x},t)=\boldsymbol{\eta}(\boldsymbol{x},t)$, though now dependent on t, remains independent of τ, as in (3.11*c*).

For any given value of τ, the $\boldsymbol{x}\mapsto\boldsymbol{x}^{\ell}(\boldsymbol{x},t,\tau)$ transformation is simply an HEL map for which the analysis of § 2 applies without modification, except for our use of ℓ rather than L. So by (2.6*a,b*), the velocity of the point $P^{\ell}:\boldsymbol{x}^{\ell}(\boldsymbol{x},t,\tau)$ at fixed $\boldsymbol{x}$ and τ is

$$\boldsymbol{w}^{*\ell}=\partial_t\boldsymbol{x}^{\ell}=\boldsymbol{w}^{\ell}\cdot\nabla\boldsymbol{x}^{\ell}\quad\text{or}\quad\boldsymbol{w}^{\ell}=\boldsymbol{w}^{*\ell}\cdot(\nabla\boldsymbol{x}^{\ell})^{-1}, \tag{3.24a,b}$$

where, of course, $\boldsymbol{w}^{*\ell}(\boldsymbol{x},t,\tau)=\boldsymbol{w}^{*}(\boldsymbol{x}^{\ell}(\boldsymbol{x},t,\tau),t)$. At $\tau=0$, where $\boldsymbol{x}^{\ell}=\boldsymbol{x}$, they simply give the Eulerian value

$$\boldsymbol{w}^{*\ell}(\boldsymbol{x},t,0)=\boldsymbol{w}^{\ell}(\boldsymbol{x},t,0)=\boldsymbol{0}, \tag{3.25}$$

while at $\tau=1$, where $\boldsymbol{x}^{\ell}=\boldsymbol{x}^{L}$, they determine the HEL values (see (2.6*a,b*))

$$\boldsymbol{w}^{*L}=\partial_t\boldsymbol{x}^{L}=\boldsymbol{w}\cdot\nabla\boldsymbol{x}^{L}\quad\text{where}\quad\boldsymbol{w}(\boldsymbol{x},t)=\boldsymbol{w}^{\ell}(\boldsymbol{x},t,1). \tag{3.26a,b}$$

The second equality in (3.24a), when written as $(\partial_t - \boldsymbol{w}^\ell \cdot \nabla)\boldsymbol{x}^\ell = \boldsymbol{0}$, may be compared with our basic identity $(\partial_\tau - \boldsymbol{\eta} \cdot \nabla)\boldsymbol{x}^\ell = \boldsymbol{0}$ (see the $n = 1$ case of (3.5b)). We differentiate the former with respect to τ and the latter with respect to t. On comparing the results and noting that $\partial_\tau \partial_t \boldsymbol{x}^\ell = \partial_t \partial_\tau \boldsymbol{x}^\ell$, we obtain

$$(\partial_t \boldsymbol{w}^\ell) \cdot \nabla \boldsymbol{x}^\ell + \boldsymbol{w}^\ell \cdot \nabla(\partial_t \boldsymbol{x}^\ell) = (\partial_\tau \boldsymbol{\eta}) \cdot \nabla \boldsymbol{x}^\ell + \boldsymbol{\eta} \cdot \nabla(\partial_\tau \boldsymbol{x}^\ell). \tag{3.27}$$

Then, using $\partial_t \boldsymbol{x}^\ell = \boldsymbol{w}^\ell \cdot \nabla \boldsymbol{x}^\ell$ and $\partial_\tau \boldsymbol{x}^\ell = \boldsymbol{\eta} \cdot \nabla \boldsymbol{x}^\ell$ again, we obtain

$$(\partial_\tau - \boldsymbol{\eta} \cdot \nabla)\boldsymbol{w}^\ell = (\partial_t - \boldsymbol{w}^\ell \cdot \nabla)\boldsymbol{\eta}, \tag{3.28a}$$

after we have post-multiplied by $(\nabla \boldsymbol{x}^\ell)^{-1}$. A more useful form of (3.28a) is

$$\partial_\tau \boldsymbol{w}^\ell = [\boldsymbol{\eta},\ \boldsymbol{w}^\ell] + \partial_t \boldsymbol{\eta}. \tag{3.28b}$$

As in § 3.3, the Taylor expansion

$$\boldsymbol{w}^\ell(\boldsymbol{x}, t, 1) = \boldsymbol{w}^\ell + \partial_\tau \boldsymbol{w}^\ell + \tfrac{1}{2}\partial_\tau^2 \boldsymbol{w}^\ell + \cdots, \tag{3.29}$$

where all variables on the right-hand side are evaluated at $\tau = 0$, determines $\boldsymbol{w}(\boldsymbol{x}, t) = \boldsymbol{w}^\ell(\boldsymbol{x}, t, 1)$ (see (3.26b)). Repeated use of (3.28b), noting (3.25), gives

$$\boldsymbol{w}(\boldsymbol{x}, t) = \partial_t \boldsymbol{\eta} + \tfrac{1}{2}[\boldsymbol{\eta},\ \partial_t \boldsymbol{\eta}] + \frac{1}{3!}[\boldsymbol{\eta},\ [\boldsymbol{\eta},\ \partial_t \boldsymbol{\eta}]] + \cdots. \tag{3.30a}$$

Its inverse derived in Appendix E is

$$\boldsymbol{w}^*(\boldsymbol{x}, t) = \partial_t \boldsymbol{\eta} - \tfrac{1}{2}[\boldsymbol{\eta},\ \partial_t \boldsymbol{\eta}] + \frac{1}{3!}[\boldsymbol{\eta},\ [\boldsymbol{\eta},\ \partial_t \boldsymbol{\eta}]] - \cdots. \tag{3.30b}$$

A comparison with (3.30a) shows that $-\boldsymbol{w}^*$ has the same expansion as $\boldsymbol{w}$ but with $\boldsymbol{\eta}$ replaced by $-\boldsymbol{\eta}$.

On combining the results (3.22b) and (3.30a), we may express the HEL advective velocity $\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{w}$ (see (2.8c)) in terms of the Eulerian velocity field $\boldsymbol{v}^*(\boldsymbol{x}, t)$ and the displacement field $\boldsymbol{\eta}(\boldsymbol{x}, t)$ in the form

$$\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{w} = \boldsymbol{v}^* - (\partial_t \boldsymbol{\eta} + [\boldsymbol{v}^*,\ \boldsymbol{\eta}]) + \tfrac{1}{2}[(\partial_t \boldsymbol{\eta} + [\boldsymbol{v}^*,\ \boldsymbol{\eta}]),\ \boldsymbol{\eta}] - \cdots. \tag{3.31a}$$

Then use of the expressions (3.22c) for $\boldsymbol{v}^*$ and (3.30b) for $\boldsymbol{w}^*$ determines the inverse

$$\boldsymbol{u}^* = \boldsymbol{v}^* - \boldsymbol{w}^* = \boldsymbol{v} + (\partial_t \boldsymbol{\eta} + [\boldsymbol{v},\ \boldsymbol{\eta}]) + \tfrac{1}{2}[(\partial_t \boldsymbol{\eta} + [\boldsymbol{v},\ \boldsymbol{\eta}]),\ \boldsymbol{\eta}] + \cdots. \tag{3.31b}$$

Finally, since $\boldsymbol{v}^*$, $\boldsymbol{v}$ and $\boldsymbol{\eta}$ are solenoidal, we have the alternative forms (3.23a,c) for $\boldsymbol{v}$ and $\boldsymbol{v}^*$. Likewise, the expansions (3.30a,b) for $\boldsymbol{w}$ and $\boldsymbol{w}^*$ may be expressed in the similar forms

$$\boldsymbol{w}(\boldsymbol{x}, t) = \partial_t \boldsymbol{\eta} - \tfrac{1}{2}\nabla \times ((\partial_t \boldsymbol{\eta}) \times \boldsymbol{\eta}) + \cdots, \tag{3.32a}$$

$$\boldsymbol{w}^*(\boldsymbol{x}, t) = \partial_t \boldsymbol{\eta} + \tfrac{1}{2}\nabla \times ((\partial_t \boldsymbol{\eta}) \times \boldsymbol{\eta}) + \cdots. \tag{3.32b}$$

4. Mean field equations

In a mean field theory we express the velocity $\boldsymbol{v}^*$ in terms of its mean $\overline{\boldsymbol{v}^*}$ and fluctuating $\boldsymbol{v}^{*\prime}$ parts and write

$$\boldsymbol{v}^* = \overline{\boldsymbol{v}^*} + \boldsymbol{v}^{*\prime}, \tag{4.1}$$

where the 'bar' is used to denote the average, while the 'prime' identifies the remaining fluctuating part. On averaging Euler's equation (1.1a), we obtain

$$\partial_t \overline{\boldsymbol{v}^*} + \overline{\boldsymbol{v}^*} \cdot \nabla \overline{\boldsymbol{v}^*} + \nabla \cdot \left(\overline{\boldsymbol{v}^{*\prime}\boldsymbol{v}^{*\prime}}\right) = -\nabla \overline{p^*}, \quad \nabla \cdot \overline{\boldsymbol{v}^*} = 0. \tag{4.2a,b}$$

The fundamental difficulty that must be faced is the evaluation of the mean Reynolds stress $\overline{\boldsymbol{v}^{*\prime}\boldsymbol{v}^{*\prime}}$. In principle, $\boldsymbol{v}^{*\prime}$ satisfies the fluctuating part of Euler's equation together with the solenoidal condition $\nabla\cdot\boldsymbol{v}^{*\prime}=0$. The determination of $\overline{\boldsymbol{v}^{*\prime}\boldsymbol{v}^{*\prime}}$ lies at the heart of all closure theories of turbulence. In this section, we consider alternative starting points based on HEL and its variants.

We stated in § 1 that we would be primarily concerned with the statistical representation of a stochastically varying flow. So from that point of view, we envisage ensemble averages but other averages, over space and time, are possible. One important space average in the geophysical context is the azimuthal average about a circle composed of points P, called a 'magic hoop' by McIntyre (1980), which are displaced into non-axisymmetric loops composed of points P^L. The magic hoop, though not its name, was first introduced by Soward (1972), who used it to elucidate the extraordinary findings of Braginsky (1964*a*, *b*). Our analysis below could be applied to such space averaging, but, since the averaging is executed about a displaced position, this Lagrangian feature, though manageable, does not really meet our objective of achieving a totally Eulerian theory.

4.1. *Generalized Lagrangian mean*

The key assumption made originally by Andrews & McIntyre (1978*a*) is that

$$\boldsymbol{u}=\overline{\boldsymbol{u}}, \tag{4.3}$$

the consequences of which are the *raison d'être* for the use of the complicated HEL formulation. Its importance lies in the fact that although $\boldsymbol{V}=\overline{\boldsymbol{V}}+\boldsymbol{V}'$ has a fluctuating part $\boldsymbol{V}'$, this does not appear in the average of $\{\boldsymbol{u},\ \boldsymbol{V}\}$, which simply becomes $\{\boldsymbol{u},\ \overline{\boldsymbol{V}}\}$. Consequently, the average of the HEL equation (2.10), namely (1.4*a*), is

$$\partial_t\overline{\boldsymbol{V}}+\{\boldsymbol{u},\ \overline{\boldsymbol{V}}\}=-\nabla\overline{\Pi}, \tag{4.4a}$$

where, from (D 7),

$$\overline{\boldsymbol{V}}=\overline{\boldsymbol{W}}+\overline{\boldsymbol{U}}=\overline{\mathbf{G}\cdot\boldsymbol{w}}+\overline{\mathbf{G}}\cdot\boldsymbol{u}. \tag{4.4b}$$

Then the average of (2.11) determines the mean field version

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\mathscr{C}}\overline{\boldsymbol{V}}\cdot\mathrm{d}\boldsymbol{x}=0 \tag{4.4c}$$

of Kelvin's circulation theorem, for a circuit $\mathscr{C}(t)$ composed of points, whose HEL coordinates $\boldsymbol{x}$ move with velocity $\boldsymbol{u}(\boldsymbol{x},t)$. Andrews & McIntyre coined the term GLM for this approach. We must emphasize at the outset that Andrews & McIntyre concentrated almost exclusively on compressible flows, with associated further thermodynamic implications.

In addition to the basic requirement $\boldsymbol{u}=\overline{\boldsymbol{u}}$ (see (4.3)), Andrews & McIntyre made the natural assumption

$$\overline{\boldsymbol{\xi}}=\mathbf{0}, \tag{4.5a}$$

on use of which the average of (2.6*a*) and (2.9*a*) determines

$$\overline{\boldsymbol{w}^{*L}}=\mathbf{0},\quad \overline{\boldsymbol{u}^{*L}}=\boldsymbol{u}, \tag{4.5b,c}$$

respectively. Since $\boldsymbol{v}^{*L}=\boldsymbol{w}^{*L}+\boldsymbol{u}^{*L}$ (see (2.9*b*)), it follows from (4.5*b*,*c*) that

$$\overline{\boldsymbol{v}^{*L}}=\boldsymbol{u}, \tag{4.5d}$$

which says that $\boldsymbol{u}(\boldsymbol{x},t)$ is the average of the fluid velocity $\boldsymbol{v}^{*L}$ at the moving HEL position $\boldsymbol{x}^L(\boldsymbol{x},t)$. Consequently, the average of (2.3*a*) may be expressed in the form

$\mathsf{p} = \boldsymbol{u} - \overline{\boldsymbol{V}} = -\overline{(\nabla\boldsymbol{\xi})\cdot\boldsymbol{v}^{*L}}$. In the context of waves riding on a mean flow, Andrews & McIntyre call p the pseudo- or wave momentum per unit mass, albeit their formula (3.1) contains an extra contribution associated with the Coriolis acceleration. Similarly, our (4.4*c*) is simply a special case of their (3.11).

The intrinsically nonlinear relation (4.5*d*) almost determines the fluctuating displacement $\boldsymbol{\xi}$. To fix it completely, we need to add one further constraint. A natural one is to demand that the mapping $\boldsymbol{x} \mapsto \boldsymbol{x}^L$ is isochoric ($\mathscr{J} \equiv \|\boldsymbol{I} + \nabla\boldsymbol{\xi}\| = 1$, see (C 2)) but this implies that

$$\nabla\cdot\boldsymbol{\xi} = \tfrac{1}{2}\nabla\cdot\big(\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi} - \boldsymbol{\xi}(\nabla\cdot\boldsymbol{\xi})\big) - \|\nabla\boldsymbol{\xi}\|. \tag{4.6}$$

Since the average of the right-hand side is generally non-zero, it follows that $\nabla\cdot\overline{\boldsymbol{\xi}} \neq 0$ too, which contradicts the assumption $\overline{\boldsymbol{\xi}} = \boldsymbol{0}$ (see (4.5*a*)). The impasse may be averted by dropping the isochoricity constraint in favour of the constraint

$$\nabla\cdot\boldsymbol{\xi} = 0. \tag{4.7}$$

Unfortunately, now that $\mathscr{J} \neq 1$, the solenoidal condition (2.13*f*) no longer holds and $\nabla\cdot\boldsymbol{u} \neq 0$. This feature is associated with the fact that wave-induced Lagrangian mean velocities generally lead to divergent Lagrangian mean particle positions even for an incompressible fluid (see McIntyre 1988 and cited references).

4.2. *GLM and its Eulerian counterpart glm*

The GLM approach just described relies on the use of the HEL Lagrangian displacement field $\boldsymbol{\xi}(\boldsymbol{x}, t)$. In this section we adopt the ETHEL formulation and work with the virtual displacement $\boldsymbol{\eta}(\boldsymbol{x}, t)$. This has the advantage that we may demand that the $\boldsymbol{x} \mapsto \boldsymbol{x}^L$ transformation is isochoric, as in § 3 and also Soward (1972), with the consequence that $\nabla\cdot\boldsymbol{\eta} = 0$ (see (3.12*a*)) and $\nabla\cdot\boldsymbol{u} = 0$ (see (2.13*f*)). We continue to make the assumption $\boldsymbol{u} = \overline{\boldsymbol{u}}$ (see (4.3)) and may then consistently assume that

$$\overline{\boldsymbol{\eta}} = \boldsymbol{0}. \tag{4.8}$$

It is, however, clear from (3.7) that $\boldsymbol{\xi}$ has a mean and fluctuating part $\boldsymbol{\xi} = \overline{\boldsymbol{\xi}} + \boldsymbol{\xi}'$ and is not solenoidal (see (3.12*b*)), just as in Soward (1972). Furthermore, since $\boldsymbol{v}^{*L} = \mathrm{D}_t\boldsymbol{x}^L$, it follows from (2.9*b*) that

$$\overline{\boldsymbol{v}^{*L}} = \boldsymbol{u} + \mathrm{D}_t\overline{\boldsymbol{\xi}}. \tag{4.9}$$

This result is unfortunate because the HEL advective velocity $\boldsymbol{u}$ is not simply $\overline{\boldsymbol{v}^{*L}}$, as it was previously (see (4.5*d*)) in the GLM approach. It means that although the mean circulation theorem (4.4*c*) continues to hold, the elegant interpretation of the HEL advective velocity involved in it as $\boldsymbol{u} = \overline{\boldsymbol{v}^{*L}}$ is lost. As mentioned at the end of § 1 and again at the beginning of § 2, we call our development based on $\overline{\boldsymbol{\eta}} = \boldsymbol{0}$, rather than $\overline{\boldsymbol{\xi}} = \boldsymbol{0}$, the ETHEL–glm approach.

When the ETHEL–glm approach is applied to stochastically varying flows, as it is in this section, its aims coincide with those of the Holm–glm approach (Holm 2002), both employing the Eulerian mean velocity $\overline{\boldsymbol{v}^{*}}$ (see (4.1)) in place of the GLM velocity $\overline{\boldsymbol{v}^{*L}}$ (see (4.5*d*)). Although agreeing on that key issue, the ETHEL–glm theory does not coincide with the Holm–glm theory. Although their starting points are the same, the ETHEL–glm theory remains completely faithful to the GLM equation (4.4*a*), while the Holm–glm theory does not. This is demonstrated in Appendix F, where the correct consequence (F 1) of the Holm–glm theory is shown to be equivalent to (4.4*a*). Although the Holm–glm equation superficially resembles (F 1), it is in fact significantly different.

Just as in the case of the GLM approach of § 4.1, where there is no restriction on the size of the Lagrangian displacement field $\boldsymbol{\xi}(\boldsymbol{x}, t)$, there is no restriction either on the size of the virtual displacement $\boldsymbol{\eta}(\boldsymbol{x}, t)$. Significantly, the assumptions $\boldsymbol{u} = \overline{\boldsymbol{u}}$ and $\overline{\boldsymbol{\eta}} = \boldsymbol{0}$ allow us to determine $\boldsymbol{u}$ and $\boldsymbol{\eta}$ from the mean and fluctuating contributions to (3.31*a*) for given $\boldsymbol{v}^*$ (see (4.12) below). Since the power series expansions (3.16*a*), (3.22*a*) and (3.31*a*) for Π, $\boldsymbol{V}$ and $\boldsymbol{u}$ are given to all orders, they provide exact solutions when the power series are convergent. The value of $\overline{\boldsymbol{V}}$ so obtained may be used in (4.4*c*) to provide an exact ETHEL–glm form of Kelvin's circulation theorem.

Although valid for $\xi = O(1)$, the GLM approach is generally used for small HEL displacements, $\xi \ll 1$, and then $\boldsymbol{\xi}$ itself, together with $\boldsymbol{u}$ and $\boldsymbol{V}$, may be obtained from expansion procedures. We therefore proceed in a similar way for the glm approach and consider expansions based on the assumption

$$\eta \ll 1. \tag{4.10}$$

We need to retain only fluctuating terms correct to $O(\eta)$ and mean terms correct to $O(\eta^2)$; in what follows the symbol $\approx$ will be used to indicate that terms smaller than these magnitudes have been neglected. So, for example, from (3.7) we obtain the fluctuating and mean contributions

$$\boldsymbol{\xi}' \approx \boldsymbol{\eta}, \quad \overline{\boldsymbol{\xi}} \approx \tfrac{1}{2}\nabla\cdot\overline{\boldsymbol{\eta}\boldsymbol{\eta}} \tag{4.11a,b}$$

to $\boldsymbol{\xi}$. The result emphasizes that $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are approximately equal but that $\boldsymbol{\xi}$ has a relatively small mean part.

As indicated above, the fluctuating and mean parts of (3.31*a*), under the assumptions $\boldsymbol{u} = \overline{\boldsymbol{u}}$ and $\overline{\boldsymbol{\eta}} = \boldsymbol{0}$, lead to

$$\boldsymbol{v}^{*\prime} \approx \partial_t\boldsymbol{\eta} + [\,\overline{\boldsymbol{v}^*},\ \boldsymbol{\eta}], \quad \boldsymbol{u} \approx \overline{\boldsymbol{v}^*} - \tfrac{1}{2}\overline{[\boldsymbol{v}^{*\prime},\ \boldsymbol{\eta}]}, \tag{4.12a,b}$$

where

$$\nabla\cdot\overline{\boldsymbol{v}^*} = 0, \quad \nabla\cdot\boldsymbol{v}^{*\prime} = 0, \quad \nabla\cdot\boldsymbol{\eta} = 0, \quad \nabla\cdot\boldsymbol{u} = 0. \tag{4.12c,d}$$

So, in principle, given $\boldsymbol{v}^{*\prime}$ and $\overline{\boldsymbol{v}^*}$, we may solve (4.12*a*), subject to some suitable initial condition for the Lie dragging velocity $\boldsymbol{\eta}$, and hence determine $\boldsymbol{u}$ from (4.12*b*). Such a technique could be extended to higher orders if needed. In the same spirit, from (3.22*a*) and (3.16*a*), we also have

$$\overline{\boldsymbol{V}} \approx \overline{\boldsymbol{v}^*} + \overline{\{\boldsymbol{\eta},\ \boldsymbol{v}^{*\prime}\}} + \tfrac{1}{2}\overline{\{\boldsymbol{\eta},\ \{\boldsymbol{\eta},\ \overline{\boldsymbol{v}^*}\}\}}, \tag{4.13a}$$

$$\overline{\Pi} \approx \overline{\Pi^*} + \overline{\boldsymbol{\eta}\cdot\nabla\Pi^{*\prime}} + \tfrac{1}{2}\overline{(\boldsymbol{\eta}\cdot\nabla)^2\overline{\Pi^*}}. \tag{4.13b}$$

From another point of view, if we know about the statistics of the displacement field $\boldsymbol{\eta}$, we may solve (4.4*a*) for $\boldsymbol{u}$ provided that we know its relation to $\overline{\boldsymbol{V}}$. The contributions to $\overline{\boldsymbol{V}} = \overline{\boldsymbol{U}} + \overline{\boldsymbol{W}}$ (see (4.4*b*)), determined by (2.6*a*), (D 6) and (D 7*c*,*e*), are

$$\overline{\boldsymbol{U}} = \overline{\boldsymbol{G}}\cdot\boldsymbol{u} \approx \boldsymbol{u} + \tfrac{1}{2}\overline{\{\boldsymbol{\eta}\,,\ \big((\nabla\boldsymbol{\eta}) + (\nabla\boldsymbol{\eta})^{\mathrm{T}}\big)\}}\cdot\boldsymbol{u}, \tag{4.14a}$$

$$\overline{\boldsymbol{W}} = \overline{\nabla\boldsymbol{x}^L\cdot\partial_t\boldsymbol{\xi}} \approx \partial_t\overline{\boldsymbol{\xi}} + \overline{(\nabla\boldsymbol{\eta})\cdot\partial_t\boldsymbol{\eta}}, \tag{4.14b}$$

where $\overline{\boldsymbol{\xi}}$ is given by (4.11*b*). Then, after the evolution of $\boldsymbol{u}$ is determined, we may recover the mean velocity $\overline{\boldsymbol{v}^*}$ from the approximate inverse of (4.12*b*), namely

$$\boldsymbol{v}^{*\prime} \approx \partial_t\boldsymbol{\eta} + [\,\boldsymbol{u},\ \boldsymbol{\eta}], \quad \overline{\boldsymbol{v}^*} \approx \boldsymbol{u} + \tfrac{1}{2}\overline{[\boldsymbol{v}^{*\prime},\ \boldsymbol{\eta}]}, \tag{4.15a,b}$$

or, more directly, simply

$$\overline{\boldsymbol{v}^*} = \overline{\boldsymbol{u}^*} + \overline{\boldsymbol{w}^*}, \tag{4.16a}$$

in which

$$\overline{\boldsymbol{u}^{*}} \approx \boldsymbol{u} + \tfrac{1}{2}\,\overline{[[\boldsymbol{u},\,\boldsymbol{\eta}],\,\boldsymbol{\eta}]}, \qquad \overline{\boldsymbol{w}^{*}} \approx \tfrac{1}{2}\,\overline{[\partial_t\boldsymbol{\eta},\,\boldsymbol{\eta}]}, \tag{4.16b,c}$$

where, in component form,

$$\overline{[[\boldsymbol{u},\,\boldsymbol{\eta}],\,\boldsymbol{\eta}]}_i = \nabla_j(\overline{\eta_j\eta_k}\,\nabla_k u_i - \overline{\eta_i\eta_k}\,\nabla_k u_j + (\,\overline{\eta_i\nabla_k\eta_j - \eta_j\nabla_k\eta_i}\,)\,u_k). \tag{4.16d}$$

Our glm formulation constitutes a way of specifying $\boldsymbol{u}$, $\overline{\boldsymbol{V}}$ and $\overline{\Pi}$ in the GLM equation (4.4a) according to the ETHEL prescription (see particularly (4.8), (4.12) and (4.13)). Roberts & Soward (2009), however, showed that the approach of Holm (2002) leads to a completely different glm equation (F 1), which Roberts & Soward (2009) called the modified Eulerian mean (MEM) equation. We outline the MEM approach in Appendix F and describe how it relates to the ETHEL formulation correct to $O(\eta^2)$.

4.3. α-*Models*

As an example of how the ETHEL–glm procedure can be implemented, we take advantage of a consequence of one of Holm's Taylor hypotheses, namely that

$$\overline{\boldsymbol{\eta}\boldsymbol{\eta}} = \alpha^2\,\boldsymbol{I}, \tag{4.17}$$

where, in general, α is constant following the mean flow velocity $\overline{\boldsymbol{v}^{*}}$. We will consider a very restricted scenario, in which α is a constant. Now Holm's method is such that he would not need to consider averages of terms involving $\partial_t\boldsymbol{\eta}$ and $\nabla\boldsymbol{\eta}$ (we say 'would' because he actually works with $\boldsymbol{\xi}$ rather than $\boldsymbol{\eta}$). In contrast, our glm method leads to expressions (4.14a,b) defining $\overline{\boldsymbol{V}} = \overline{\boldsymbol{U}} + \overline{\boldsymbol{W}}$, in which such terms abound and the average $\overline{\boldsymbol{\eta}\boldsymbol{\eta}}$ is completely absent. So to make any contact with Holm's development, we make the strong assumption that the averages of terms involving $\partial_t\boldsymbol{\eta}$ and $\nabla\boldsymbol{\eta}$ simply vanish, implying that $\overline{\boldsymbol{U}} \approx \boldsymbol{u}$ and $\overline{\boldsymbol{W}} \approx \boldsymbol{0}$. Then Euler's equation (4.4a) becomes

$$\partial_t\boldsymbol{u} + \{\boldsymbol{u},\,\boldsymbol{u}\} + \nabla\overline{\Pi} = \boldsymbol{0}. \tag{4.18a}$$

Interestingly, $\overline{\boldsymbol{\eta}\boldsymbol{\eta}}$ does appear in the relation between the Lagrangian mean velocity $\boldsymbol{u}$ and the Eulerian mean velocity $\overline{\boldsymbol{v}^{*}}$ through the term (4.16d). Indeed, use of (4.16a–d) shows that $\overline{\boldsymbol{v}^{*}} \approx \boldsymbol{u} + (\alpha^2\nabla^2\boldsymbol{u}/2)$, or equivalently to the same order of accuracy

$$\boldsymbol{u} \approx \overline{\boldsymbol{v}^{*}} - \tfrac{1}{2}\,\alpha^2\,\nabla^2\overline{\boldsymbol{v}^{*}}, \qquad \nabla\cdot\boldsymbol{u} = 0. \tag{4.18b,c}$$

The system (4.18a–c) achieves our objective of deriving Eulerian mean equations for $\overline{\boldsymbol{v}^{*}}$. From this point of view, the equations still involve the α-term and the Helmholtz equation structure is still present in (4.18b). Nevertheless, it is striking that (4.18a), governing the evolution of $\boldsymbol{u}$, is independent of $\overline{\boldsymbol{v}^{*}}$.

Interestingly, based on the above restricted scenario, Roberts & Soward (2009) showed that the MEM approach, mentioned at the end of § 4.2, leads to

$$\partial_t\overline{\boldsymbol{V}^{E}} + \tfrac{1}{2}\{\overline{\boldsymbol{v}^{*}},\,\overline{\boldsymbol{V}^{E}}\} + \tfrac{1}{2}\{\overline{\boldsymbol{V}^{E}},\,\overline{\boldsymbol{v}^{*}}\} - \tfrac{1}{2}\alpha^2\,\nabla^2\,\{\overline{\boldsymbol{v}^{*}},\,\overline{\boldsymbol{v}^{*}}\} = -\nabla\overline{\Pi^{E}} \tag{4.19a}$$

(but see also Appendix F), where

$$\overline{\boldsymbol{V}^{E}} \approx \overline{\boldsymbol{v}^{*}} - \alpha^2\,\nabla^2\overline{\boldsymbol{v}^{*}}, \qquad \nabla\cdot\overline{\boldsymbol{V}^{E}} \approx 0. \tag{4.19b,c}$$

Holm (2002) referred to $\overline{\boldsymbol{v}^{*}}$ as the 'filtered' velocity and $\overline{\boldsymbol{V}^{E}}$ as the 'unfiltered' velocity, because $\overline{\boldsymbol{v}^{*}}$ is smoother than $\overline{\boldsymbol{V}^{E}}$. Since they are related by the Helmholtz equation (4.19b), Geurts *et al.* (2008, equation (4)) called $\alpha = O(\eta)$ the 'Helmholtz-length'. Furthermore, since $\boldsymbol{u} = \overline{\boldsymbol{v}^{*}} + O(\eta)$ and $\overline{\boldsymbol{V}^{E}} = \overline{\boldsymbol{v}^{*}} + O(\eta)$, the zeroth-order approximation of the mean Euler's equation (F 1) determines the relation

$$-\alpha^2 \nabla^2 \{\overline{\boldsymbol{v}^*},\ \overline{\boldsymbol{v}^*}\} \approx \partial_t(\alpha^2 \nabla^2 \overline{\boldsymbol{v}^*}) + \nabla(\alpha^2 \nabla^2 \overline{\Pi^E}). \tag{4.20}$$

Since $\overline{\boldsymbol{V}^E} \approx \boldsymbol{u} - (\alpha^2\nabla^2\overline{\boldsymbol{v}^*}/2)$ (see (4.18*b*) and (4.19*b*)), use of (4.20) establishes the equivalence of (4.18*a*) and (4.19*a*). Obviously, the complexity of (4.19*a*), governing the evolution of $\overline{\boldsymbol{V}^E}$, is misleading and emphasizes the less satisfactory nature of the MEM–glm approach. So arguably, the pursuit of the ETHEL–glm equation (4.18*a*), governing the evolution of $\boldsymbol{u}$, is preferable in the Holm–α context. Nevertheless, the bald nature of (4.18*a*), whose structure is similar in spirit to omitting the crucial mean Reynolds stress $\overline{\boldsymbol{v}^{*\prime}\boldsymbol{v}^{*\prime}}$ term in (4.2*a*), suggests that less restrictive closure assumptions are needed to obtain useful mean field equations.

5. Concluding remarks

It should be apparent that the programme laid out in § 1 is now complete. The GLM method for incorporating the effect of waves on mean flow has been successfully recast into Eulerian (glm) form while preserving the significant conservation law known as Kelvin's theorem. At the outset, the relationship between the position $P^L : \boldsymbol{x}^L \equiv \boldsymbol{x} + \boldsymbol{\xi}(\boldsymbol{x}, t)$ of a fluid element displaced from its mean position $P : \boldsymbol{x}$ by the waves was described as a mapping, $\boldsymbol{x} \mapsto \boldsymbol{x}^L$. The HEL formalism was then invoked in § 2. This links flow properties at P^L, such as velocity and pressure, to the mean position P and led to the basic GLM consequence (4.4*a*) of Euler's equation, namely

$$\partial_t \overline{\boldsymbol{V}} + \mathrm{L}_{\boldsymbol{u}} \overline{\boldsymbol{V}} = -\nabla \overline{\Pi}, \tag{5.1}$$

involving the Lie derivative $\mathrm{L}_{\boldsymbol{u}}\overline{\boldsymbol{V}} = \{\boldsymbol{u},\ \overline{\boldsymbol{V}}\}$. Although Kelvin's theorem follows easily from (5.1), it has the disadvantage of describing flow properties at P^L in terms of the different point P and the next task was therefore to recast (5.1) in Eulerian terms. For this purpose, the ETHEL technique of § 3 was developed based on Lie dragging.

Previous attempts to develop an ETHEL formalism have been based on an expansion to second order in ξ and have paid a heavy price: Kelvin's theorem has been lost in the process (see Soward & Roberts 2008 and Roberts & Soward 2009). The Lie dragging recipe employed in § 3 provides the natural representation of the HEL variables in terms of Taylor expansions involving the appropriate Lie derivative L_η of the Eulerian variables. In this way, expansions were constructed in terms of the dragging velocity $\boldsymbol{\eta}^*$, introduced by Moffatt (1986), rather than the displacement $\boldsymbol{\xi}$. The advantage gained by ETHEL lies in the simple structure of the Lie-derivative Taylor expansions, which permit the ETHEL procedure to be taken to all orders in η.

Superficially, the link (3.11*d*) between the displacement $\boldsymbol{\xi}$ and the contravariant Lie dragging velocity $\boldsymbol{\eta}\,(= \boldsymbol{\eta}^*)$ is direct and succinct:

$$\boldsymbol{\eta}(\boldsymbol{x} + \boldsymbol{\xi}, t) = \boldsymbol{\eta}(\boldsymbol{x}, t) + \boldsymbol{\eta}(\boldsymbol{x}, t) \cdot \nabla \boldsymbol{\xi}(\boldsymbol{x}, t). \tag{5.2a}$$

This, however, conceals some difficulties, particularly the lack of uniqueness of the solution of (5.2*a*). We have overcome this by tying the solution of (5.2*a*) to the expansion

$$\boldsymbol{\xi} = \boldsymbol{\eta} + \tfrac{1}{2}\boldsymbol{\eta} \cdot \nabla \boldsymbol{\eta} + \cdots + \frac{1}{n!}(\boldsymbol{\eta} \cdot \nabla)^{n-1} \boldsymbol{\eta} + \cdots \tag{5.2b}$$

(see (3.7)), which, like any other Taylor expansion, is arguably convergent for all sufficiently small η and which tends to $\boldsymbol{\eta}$ for $\eta \to 0$. This solution may be checked directly from (5.2*a*) upon use of the Taylor series expansion

$$\boldsymbol{\eta}(\boldsymbol{x}+\boldsymbol{\xi},t)=\boldsymbol{\eta}+\boldsymbol{\xi}\cdot\nabla\boldsymbol{\eta}+\tfrac{1}{2}(\boldsymbol{\xi}\boldsymbol{\xi}):(\nabla\nabla)\,\boldsymbol{\eta}+\frac{1}{3!}(\boldsymbol{\xi}\boldsymbol{\xi}\boldsymbol{\xi})\vdots(\nabla\nabla\nabla)\,\boldsymbol{\eta}+\cdots \tag{5.2c}$$

(in dyadic notation $(\boldsymbol{\xi}\boldsymbol{\xi}):(\nabla\nabla)\equiv(\xi_i\xi_j)(\nabla_i\nabla_j)$, etc.). Following the substitution of (5.2*b*) into (5.2*c*), the verification of the cancellation of the terms in (5.2*a*) at various orders of η is a cumbersome task, which after $O(\eta^4)$ (as displayed in (5.2*c*)) becomes increasingly laborious. Similarly, the relation (5.2*b*) may be inverted awkwardly by an iterative procedure:

$$\boldsymbol{\eta}=\boldsymbol{\xi}-\tfrac{1}{2}\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi}+\left[\tfrac{1}{12}(\boldsymbol{\xi}\cdot\nabla)^2\boldsymbol{\xi}+\tfrac{1}{4}(\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi})\cdot\nabla\boldsymbol{\xi}\right]+\cdots. \tag{5.2d}$$

Although (5.2*d*) fails to have the elegant structure of (5.2*b*), it does provide a recipe whereby $\boldsymbol{\eta}$ is defined uniquely by $\boldsymbol{\xi}$. Furthermore, like (5.2*b*), (5.2*d*) is arguably convergent for sufficiently small ξ, although this is less clear since the $O(\xi^n)$ term is not obtained readily.

The cumbersome nature of (5.2*d*) was not apparent in the work of Moffatt (1986), where $O(\xi^3)$ terms were neglected. Indeed, the result (5.2*d*) exposes the drawbacks of using the HEL displacement $\boldsymbol{\xi}$, which is neither an Eulerian nor an HEL variable. Instead we find that the HEL Lie dragging velocity $\boldsymbol{\eta}$, or its Eulerian equivalent $\boldsymbol{\eta}^*\,(=\boldsymbol{\eta})$, provides the natural expansion variable. So, for example, the expansion, similar to (5.2*b*), of the Eulerian velocity $\boldsymbol{v}^*$ in terms of the contravariant HEL variables $\boldsymbol{v}$ and $\boldsymbol{\eta}$ obtained from (3.22*c*) is

$$\boldsymbol{v}^*(\boldsymbol{x},t)=\boldsymbol{v}+[\boldsymbol{v},\,\boldsymbol{\eta}]+\tfrac{1}{2}[[\boldsymbol{v},\,\boldsymbol{\eta}],\,\boldsymbol{\eta}]+\cdots. \tag{5.3}$$

Likewise, the expansion of the HEL velocities $\boldsymbol{V}$ and $\boldsymbol{u}$ in terms of the Eulerian variables $\boldsymbol{v}^*$ and $\boldsymbol{\eta}^*\,(=\boldsymbol{\eta})$ obtained from (3.22*a*) and (3.31*a*) are

$$\boldsymbol{V}(\boldsymbol{x},t)=\boldsymbol{v}^*+\{\boldsymbol{\eta}^*,\,\boldsymbol{v}^*\}+\tfrac{1}{2}\{\boldsymbol{\eta}^*,\,\{\boldsymbol{\eta}^*,\,\boldsymbol{v}^*\}\}+\cdots, \tag{5.4a}$$

$$\boldsymbol{u}(\boldsymbol{x},t)=\boldsymbol{v}^*-(\partial_t\boldsymbol{\eta}^*+[\boldsymbol{v}^*,\,\boldsymbol{\eta}^*])+\tfrac{1}{2}[(\partial_t\boldsymbol{\eta}^*+[\boldsymbol{v}^*,\,\boldsymbol{\eta}^*]),\,\boldsymbol{\eta}^*]-\cdots. \tag{5.4b}$$

Any attempt to use $\boldsymbol{\xi}$ instead leads to the complications encountered in (5.2*d*). In so far as the series (5.4*a*,*b*) are convergent, they give exact expressions for $\boldsymbol{V}$ and $\boldsymbol{u}$. With $\boldsymbol{\eta}^*\,(=\boldsymbol{\eta})$ chosen at our convenience, initial data at time $t=0$ (say) for $\boldsymbol{v}^*$ may be used to determine initial data for $\boldsymbol{V}$ and $\boldsymbol{u}$. Their temporal evolution is then determined from the HEL Euler equation (5.1). At any later time, $\boldsymbol{v}^*$ may be recovered from (5.3) supplemented by $\boldsymbol{v}=\boldsymbol{u}+\boldsymbol{w}$, where, from (3.30*a*),

$$\boldsymbol{w}(\boldsymbol{x},t)=\partial_t\boldsymbol{\eta}^*+\tfrac{1}{2}[\boldsymbol{\eta}^*,\,\partial_t\boldsymbol{\eta}^*]+\cdots. \tag{5.5}$$

As we have not strayed from the HEL framework, the HEL version (2.11) of Kelvin's circulation theorem continues to hold. The recipe just outlined explains how the ETHEL ought to be implemented but evades the matter of how to determine the $\boldsymbol{\eta}$ necessary to determine the complicated link $\boldsymbol{V}=\mathbf{G}\cdot(\boldsymbol{u}+\boldsymbol{w})$ (see (D 7*a*)) between $\boldsymbol{V}$ and $\boldsymbol{u}$. A mean field route through all these complex issues was outlined in § 4.

ETHEL was developed to apply to systems in which the mapping $\boldsymbol{x}\mapsto\boldsymbol{x}^L$ is determined by statistical considerations and in which what we call an ETHEL–glm theory can be created. Two examples have been mentioned: the problem of waves on a mean flow considered by Andrews & McIntyre (1978*a*) and the aim of Holm (2002) to derive the NS–α equation (or something like it) from the Euler equation. The second of these also requires a link between $\boldsymbol{V}$ and $\boldsymbol{u}$, such as a Taylor hypothesis of the type proposed by Holm (2002). Both have to divide $\boldsymbol{V}$, $\boldsymbol{u}$ and Π into mean and fluctuating parts, e.g. $\boldsymbol{V} = \overline{\boldsymbol{V}} + \boldsymbol{V}'$. As already mentioned, $\boldsymbol{u}$ describes the advective action of the mean flow and it is therefore natural to assume, as both Andrews & McIntyre and Holm did, that $\boldsymbol{u}' = \boldsymbol{0}$ so that $\boldsymbol{u} = \overline{\boldsymbol{u}}$. The additional GLM assumption $\overline{\boldsymbol{\xi}} = \boldsymbol{0}$ has the attraction in the HEL setting that $\overline{\boldsymbol{v}^{*L}} = \boldsymbol{u}$ (see (4.9)) as well, but has the unfortunate consequence, especially for an incompressible fluid that, in general $\nabla \cdot \boldsymbol{u} \neq 0$, i.e. we cannot impose that the mapping $\boldsymbol{x} \mapsto \boldsymbol{x}^L$ is isochoric. This is the well-known divergence effect about which much has been written (see e.g. McIntyre 1980, 1988). The alternative formulation through $\boldsymbol{\eta}$ enjoys a new advantage: there seems to be no obvious objection to the assumption $\overline{\boldsymbol{\eta}} = \boldsymbol{0}$ and this, together with $\overline{\boldsymbol{u}} = \boldsymbol{u}$, appears to make the $\boldsymbol{x} \to \boldsymbol{x}^L$ mapping unique:

$$\overline{\boldsymbol{u}} = \boldsymbol{u}, \quad \overline{\boldsymbol{\eta}} = \boldsymbol{0}, \tag{5.6a,b}$$

with the attractive consequence that the mean velocity $\boldsymbol{u}$ remains solenoidal, $\nabla \cdot \boldsymbol{u} = 0$, for our incompressible flow. Of course, ETHEL–glm variables differ from GLM variables because of the distinctive assumptions $\overline{\boldsymbol{\eta}} = \boldsymbol{0}$ and $\overline{\boldsymbol{\xi}} = \boldsymbol{0}$ made, respectively, in each theory. This distinction is brought out forcefully by the fact that generally $\overline{\boldsymbol{v}^{*L}} \neq \boldsymbol{u}$ in the ETHEL–glm theory, but being an Eulerian theory it is hardly surprising that a Lagrangian feature of GLM is lost.

Previous attempts to generate the NS–α equation have also truncated the formalism at $O(\xi^2)$. In the slightly different MEM approach of Roberts & Soward (2009), the Taylor ansatz $\overline{\boldsymbol{\xi}\boldsymbol{\xi}} = \alpha^2 \boldsymbol{I}$ gave

$$\partial_t \overline{\boldsymbol{V}^E} + \tfrac{1}{2}\{\overline{\boldsymbol{v}^*},\, \overline{\boldsymbol{V}^E}\} + \tfrac{1}{2}\{\overline{\boldsymbol{V}^E},\, \overline{\boldsymbol{v}^*}\} - \tfrac{1}{2}\alpha^2 \nabla^2 \{\overline{\boldsymbol{v}^*},\, \overline{\boldsymbol{v}^*}\} = -\nabla \overline{\Pi^E} \tag{5.7a}$$

(see (4.19a)), where $\overline{\boldsymbol{V}^E}$ is Holm's unfiltered velocity (4.19b)). The ETHEL–glm theory can be similarly truncated at $O(\eta^2)$, and the Taylor ansatz $\overline{\boldsymbol{\eta}\boldsymbol{\eta}} = \alpha^2 \boldsymbol{I}$, as shown in §4.3, then gives

$$\partial_t \boldsymbol{u} + \{\boldsymbol{u},\, \boldsymbol{u}\} + \nabla \overline{\Pi} = \boldsymbol{0}. \tag{5.7b}$$

Since $\boldsymbol{\eta} \sim \boldsymbol{\xi}$ for $\xi \to 0$, (5.7a) and (5.7b) should be completely equivalent from an asymptotic point of view, and Appendix F is devoted to showing that this is true. Nevertheless, (5.7b) together with (4.4c) implies the Kelvin theorem

$$\frac{\mathrm{d}}{\mathrm{d}t} \oint_{\mathscr{C}} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x} = 0, \tag{5.7c}$$

whereas there is no corresponding Kelvin theorem obtainable from (5.7a). It is now apparent that this is because (5.7a) invites a fruitless attempt to seek conservation of the circulation of $\overline{\boldsymbol{V}^E}$ round a circuit moving with velocity $\overline{\boldsymbol{v}^*}$ when, in reality, as (5.7c) shows, it is the circulation of $\boldsymbol{u}$ round a circuit $\mathscr{C}(t)$ moving with velocity $\boldsymbol{u}$ that is actually conserved.

Although this clarifies an issue left obscure by Soward & Roberts (2008) and Roberts & Soward (2009), and although (5.7b) is the correct form of the NS–α theory, at least within the framework of the same Taylor hypotheses that has been used in previous attempts to derive the NS–α equations, it seems doubtful whether (5.7b) could play the same useful role in large-eddy simulations as has been played by the existing NS–α theory. Indeed, even though (5.7b) and (5.7a) are asymptotically equivalent, their numerical simulation in turbulence applications will be different. Our claim is not that we have improved on the existing NS–α theory but rather that we have developed the ETHEL framework that is best suited to Eulerian formulations of HEL. Detailed and forceful applications of ETHEL are outside the scope of the present paper and must be left for future research.

We reiterate that we have demonstrated the utility of $\boldsymbol{\eta}$, showing how expansions in $\boldsymbol{\eta}$ can be carried out to all orders (see particularly (5.3)–(5.5)) in sharp contrast to analogous expansions in $\boldsymbol{\xi}$, which, after the few orders, become too unwieldy to continue further (cf. (5.2*b*) and (5.2*d*)). Whether or not ETHEL is more useful in practice than HEL remains to be seen. Be that as it may, the main objective has been, as the name ETHEL suggests, an Eulerian development in which physical (dependent) variables are defined at the position P of the independent variable $\boldsymbol{x}$ and not at some displaced position P^L.

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Appendix A. Relation to general tensors

The development of the HEL and the ETHEL theory in this paper follows the traditional GLM theory lines based on the mapping of a point $P : \boldsymbol{x}$ to another point $P^L : \boldsymbol{x}^L$ as introduced in (1.2). So when we consider the velocity $\boldsymbol{v}^*$ evaluated at P^L rather than P, we call it $\boldsymbol{v}^{*L}$ (see (2.2*a*)). As is well known in the GLM theory, the alternative representations $\boldsymbol{V} \equiv (\nabla \boldsymbol{x}^L) \cdot \boldsymbol{v}^{*L}(\boldsymbol{x}, t)$ and $\boldsymbol{v} \equiv \boldsymbol{v}^{*L}(\boldsymbol{x}, t) \cdot (\nabla \boldsymbol{x}^L)^{-1}$ (see (2.3*a*,*b*)) are far more useful. These are the forms that arise naturally in the theory of general tensors, which relies on a slightly different conceptual basis.

In a general tensor formulation (see Roberts & Soward 2006*b*), the point P^L in the fluid may be identified in a Cartesian frame $\mathscr{F}^L$ by coordinates x^{Li}. The coordinates x^i of $\boldsymbol{x}$ that define $\boldsymbol{x}^L(\boldsymbol{x})$ are then regarded as non-Cartesian coordinates of P^L in a reference frame $\mathscr{F}$. For the sake of clarity, we suppress the dependence on t, which is irrelevant to our concerns here. From this point of view the components of the velocity $\boldsymbol{v}^*$ and modified pressure gradient $\nabla \Pi^*$ at the point P^L in the reference frame $\mathscr{F}$ have covariant (lower index) and contravariant (upper index) components

$$v_i \equiv (\boldsymbol{V})_i, \quad v^i \equiv (\boldsymbol{v})_i \quad \text{and} \quad \Pi_{,i} \equiv (\nabla \Pi)_i, \tag{A 1a,b,c}$$

where, in the Cartesian frame adopted in this paper, $(\boldsymbol{V})_i$ and $(\boldsymbol{v})_i$ are the ith components of the vectors $\boldsymbol{V}$ and $\boldsymbol{v}$ defined by (2.3*a*,*b*). In contrast, the covariant and contravariant components of $\boldsymbol{v}^*$ at the point P^L in the Cartesian frame $\mathscr{F}^L$ are identical and simply $(\boldsymbol{v}^{*L})_i$ as defined by (2.2*a*,*b*), while the corresponding Cartesian components of $\nabla \Pi^*$ are $\partial \Pi^{*L}/\partial x^{Li}$. A further consequence pertains to the scalar $\nabla \cdot \boldsymbol{v}^*$ evaluated in $\mathscr{F}^L$ at the point P^L, namely $(\nabla \cdot \boldsymbol{v}^*)^L$, which, according to (C 1*a*), is

$$\mathscr{J}^{-1}(\mathscr{J} v^i)_{,i} \equiv \mathscr{J}^{-1} \nabla \cdot (\mathscr{J} \boldsymbol{v}) \quad \text{in} \quad \mathscr{F}, \tag{A 2}$$

where $\mathscr{J}$ is the Jacobian of the transformation $\boldsymbol{x} \mapsto \boldsymbol{x}^L$ (see Appendix C). These are the essential general tensor concepts that relate to our Cartesian development, in which $\boldsymbol{x}$ at P and $\boldsymbol{x}^L$ at P^L are distinct points in the fluid.

If, however, as in § 3 we introduce curves composed of points $P^\ell(\tau) : x^{\ell i}(\boldsymbol{x}, \tau)$ $(0 \leqslant \tau \leqslant 1)$ (see (3.1)) such that $P^\ell(0) = P$ and $P^\ell(1) = P^L$, the velocity vector $\boldsymbol{v}^*$ at

P at $\tau = 0$ may be dragged along (see e.g. d'Inverno 1992) the curve to obtain, by complete analogy with (A 1*a*,*b*), the covariant and contravariant components

$$v_i^\ell \equiv (\boldsymbol{V}^\ell)_i \quad \text{and} \quad v^{\ell i} \equiv (\boldsymbol{v}^\ell)_i, \tag{A 3a,b}$$

respectively at position P^ℓ, where, in the Cartesian frame adopted in the paper, $(\boldsymbol{V}^\ell)_i$ and $(\boldsymbol{v}^\ell)_i$ are the ith components of the vectors $\boldsymbol{V}^\ell$ and $\boldsymbol{v}^\ell$ defined by (3.17*a*,*b*). The essential point here is that for any given τ, the coordinate frame $\mathscr{F}^\ell$, in which $x^{\ell i}$ lie, is Cartesian. This means that the nature of the coordinate frame $\mathscr{F}$ is dependent on the value of τ. Thus, when $\tau = 0$, the coordinates $x^i \in \mathscr{F}$ are Cartesian with $v^{\ell i} = v_i^\ell = (\boldsymbol{v}^*)_i$, but they are non-Cartesian when $\tau > 0$. Indeed, when $\tau = 1$, $v^{\ell i} = v^i$ and $v_i^\ell = v_i$, allowing for the alternative interpretation of v^i and v_i as the dragged-along vectors (see d'Inverno 1992, § 6.2, particularly Fig. 6.3), i.e. our use of the Cartesian coordinates $(\boldsymbol{x})_i$ of the position P as HEL coordinates for P^L has established usage in the dragged-along sense.

With the exception only of this appendix, we have adopted the Cartesian interpretation of $\boldsymbol{x}$. Then $\boldsymbol{V}$ and $\boldsymbol{v}$ are simply distinct vectors with magnitudes and directions at P, which relate to the fluid velocity elsewhere at P^L. This has the advantage that classical vector manipulation may be undertaken unambiguously. However, when correctly interpreted, our analysis may be cast in the general tensor framework. It is, therefore, helpful to use general tensor terminology to identify the appropriate general tensor analogues.

Appendix B. Useful transformation properties

In this appendix and the appendices that follow, unlike Appendix A, we revert to the development elsewhere in this paper, where the vectors $\boldsymbol{x}$ and $\boldsymbol{x}^L$ possessing Cartesian coordinates (x_1, x_2, x_3) and (x_1^L, x_2^L, x_3^L) define distinct points P and P^L, respectively, as explained in the vicinity of (2.2) and (2.3) of § 2. Consider the arbitrary vector field $\boldsymbol{f}^*(\boldsymbol{x}, t)$ and (appealing to appropriate general tensor terminology, explained in Appendix A) its covariant and contravariant HEL counterparts

$$\boldsymbol{F}(\boldsymbol{x}, t) \equiv (\nabla \boldsymbol{x}^L) \cdot \boldsymbol{f}^{*L}(\boldsymbol{x}, t), \quad \boldsymbol{f}^{*L}(\boldsymbol{x}, t) \equiv \boldsymbol{f}(\boldsymbol{x}, t) \cdot \nabla \boldsymbol{x}^L, \tag{B 1a,b}$$

together with $\boldsymbol{v}^*(\boldsymbol{x}, t)$ and its contravariant HEL counterpart $\boldsymbol{v}(\boldsymbol{x}, t)$ (see (2.3*b*)). The Lie derivative $\mathrm{L}_{\boldsymbol{v}}$ of $\boldsymbol{F}$ and $\boldsymbol{f}$ may be expressed in terms of the bilinear operators $\{\boldsymbol{v},\ \}$ and $[\boldsymbol{v},\ ]$ defined by

$$\mathrm{L}_{\boldsymbol{v}} \boldsymbol{F} \equiv \{\boldsymbol{v}, \boldsymbol{F}\} \equiv \boldsymbol{v} \cdot \nabla \boldsymbol{F} + (\nabla \boldsymbol{v}) \cdot \boldsymbol{F}, \tag{B 2a}$$

$$\mathrm{L}_{\boldsymbol{v}} \boldsymbol{f} \equiv [\boldsymbol{v},\ \boldsymbol{f}] \equiv \boldsymbol{v} \cdot \nabla \boldsymbol{f} - \boldsymbol{f} \cdot \nabla \boldsymbol{v}, \tag{B 2b}$$

where, in component form, $\big((\nabla \boldsymbol{v}) \cdot \boldsymbol{F}\big)_i \equiv (\nabla_i v_j) F_j$ are important because they respectively possess the covariant and contravariant transformation properties

$$\{\boldsymbol{v},\ \boldsymbol{F}\} = (\nabla \boldsymbol{x}^L) \cdot \{\boldsymbol{v}^*,\ \boldsymbol{f}^*\}^L, \quad [\boldsymbol{v}^*,\ \boldsymbol{f}^*]^L = [\boldsymbol{v},\ \boldsymbol{f}] \cdot \nabla \boldsymbol{x}^L, \tag{B 2c,d}$$

which are established by application of (2.4*a*,*b*) using $\psi^* = v_i^*$ and f_i^*.

We now consider the transformation properties of time derivatives. To this end we need the gradient of the identity $(\partial_t - \boldsymbol{w} \cdot \nabla)\, \boldsymbol{x}^L = \boldsymbol{0}$ (see (2.6*c*)), namely

$$(\partial_t - \boldsymbol{w} \cdot \nabla)\, (\nabla \boldsymbol{x}^L) = (\nabla \boldsymbol{w}) \cdot (\nabla \boldsymbol{x}^L) \tag{B 3}$$

(see Roberts & Soward 2006*a*, equation (3*a*,*b*)). For the choice $\psi^* = f_i^*$, (2.7) gives, with the help of (2.6*b*) and (B 3), the results

$$(\nabla \boldsymbol{x}^L)\cdot(\partial_t \boldsymbol{f}^*)^L = \partial_t \boldsymbol{F} - \{\boldsymbol{w},\ \boldsymbol{F}\}, \tag{B 4a}$$

$$(\partial_t \boldsymbol{f}^*)^L = (\partial_t \boldsymbol{f} - [\boldsymbol{w},\ \boldsymbol{f}])\cdot\nabla \boldsymbol{x}^L \tag{B 4b}$$

(see Roberts & Soward 2006*b*, equations (6*b*,*d*) for the general tensor forms). The latter has the intriguing consequence that when $\boldsymbol{f}^* = \boldsymbol{w}^*$ (implying $\boldsymbol{f} = \boldsymbol{w}$),

$$(\partial_t \boldsymbol{w}^*)^L = (\partial_t \boldsymbol{w})\cdot(\nabla \boldsymbol{x}^L), \tag{B 5a}$$

since $\mathrm{L}_{\boldsymbol{w}}\boldsymbol{w} \equiv [\boldsymbol{w},\ \boldsymbol{w}] = \boldsymbol{0}$. We emphasize that this result is peculiar to the choice $\boldsymbol{f}^* = \boldsymbol{w}^*$, as, in general, $\mathrm{L}_{\boldsymbol{w}}\boldsymbol{f} \neq \boldsymbol{0}$ with the consequence

$$(\partial_t \boldsymbol{f}^*)^L \neq (\partial_t \boldsymbol{f})\cdot(\nabla \boldsymbol{x}^L), \quad \text{when} \quad \boldsymbol{f}^* \neq \boldsymbol{w}^*. \tag{B 5b}$$

Finally, on adding (B 2*c*,*d*) to (B 4*a*,*b*), we obtain

$$(\nabla \boldsymbol{x}^L)\cdot(\partial_t \boldsymbol{f}^* + \{\boldsymbol{v}^*,\ \boldsymbol{f}^*\})^L = \partial_t \boldsymbol{F} + \{\boldsymbol{u},\ \boldsymbol{F}\}, \tag{B 6a}$$

$$(\partial_t \boldsymbol{f}^* + [\boldsymbol{v}^*,\ \boldsymbol{f}^*])^L = (\partial_t \boldsymbol{f} + [\boldsymbol{u},\ \boldsymbol{f}])\cdot\nabla \boldsymbol{x}^L, \tag{B 6b}$$

respectively, in which $\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{w}$ (see (2.8*c*)).

Appendix C. The solenoidal vectors $\boldsymbol{v}$, $\boldsymbol{w}$ and $\boldsymbol{u}$

In order to address the solenoidal condition (1.1*b*), we note that $\nabla\cdot\boldsymbol{v}^*$ may be expressed in HEL form by use of standard identities (see e.g. Roberts & Soward 2006*a*, equation (13*b*)):

$$(\nabla\cdot\boldsymbol{v}^*)^L = \mathscr{J}^{-1}\nabla\cdot(\mathscr{J}\boldsymbol{v}), \tag{C 1a}$$

where $\mathscr{J} \equiv \|\nabla \boldsymbol{x}^L\|$ is the Jacobian of the transformation $\boldsymbol{x} \mapsto \boldsymbol{x}^L$, which satisfies

$$\partial_t \mathscr{J} = \nabla\cdot(\mathscr{J}\boldsymbol{w}) \tag{C 1b}$$

(see e.g. Roberts & Soward 2006*a*, equation (6)).

When the map is isochoric

$$\mathscr{J} \equiv \|\nabla \boldsymbol{x}^L\| = 1, \tag{C 2}$$

and $\boldsymbol{v}^*$ is solenoidal, (C 1*a*) implies that $\nabla\cdot\boldsymbol{v} = 0$ and (C 1*b*) implies that $\nabla\cdot\boldsymbol{w} = 0$. Since $\boldsymbol{w}^*$, like $\boldsymbol{v}^*$, satisfies the relation (C 1*a*), we also have $\nabla\cdot\boldsymbol{w}^* = 0$. These consequences are summarized by (2.13*a*–*f*).

Appendix D. The symmetric tensor G

The symmetric covariant tensor $G_{ij} \equiv (\nabla \boldsymbol{x}^L)_{ik}(\nabla \boldsymbol{x}^L)_{jk}$ may be expressed compactly in the form

$$\mathsf{G} \equiv (\nabla \boldsymbol{x}^L)\cdot(\nabla \boldsymbol{x}^L)^{\mathrm{T}}, \tag{D 1}$$

where the superscript $^{\mathrm{T}}$ denotes the transpose of the matrix. In terms of $\boldsymbol{x}^\ell(\boldsymbol{x}, \tau)$ (see (3.1*a*)), we define

$$\mathsf{G}^\ell(\boldsymbol{x}, \tau) \equiv (\nabla \boldsymbol{x}^\ell)\cdot(\nabla \boldsymbol{x}^\ell)^{\mathrm{T}}, \tag{D 2a}$$

with the properties

$$\mathsf{G}^\ell(\boldsymbol{x}, 0) = \mathsf{I}, \quad \mathsf{G}^\ell(\boldsymbol{x}, 1) = \mathsf{G}, \tag{D 2b,c}$$

implied by (3.1*b*,*c*).

With the help of

$$(\partial_\tau - \boldsymbol{\eta}\cdot\nabla)(\nabla\boldsymbol{x}^\ell) = (\nabla\boldsymbol{\eta})\cdot(\nabla\boldsymbol{x}^\ell), \tag{D 3}$$

namely the τ-time version of (B 3), direct evaluation of $(\partial_\tau - \boldsymbol{\eta}\cdot\nabla)\,\mathbf{G}^\ell$ yields the Lie derivative

$$\mathrm{L}_{\boldsymbol{\eta}}\mathbf{G}^\ell \equiv \partial_\tau\,\mathbf{G}^\ell = \{\boldsymbol{\eta},\ \mathbf{G}^\ell\} \tag{D 4a}$$

of any symmetric second rank covariant tensor (see d'Inverno 1992, equation (6.17) and cf. Roberts & Soward 2006*a*, equation (A.15*b*) with (3*a*)), where

$$\{\boldsymbol{\eta}, \bullet\} \equiv \boldsymbol{\eta}\cdot\nabla\ \bullet + (\nabla\boldsymbol{\eta})\cdot\bullet\ +\ \bullet\cdot\ (\nabla\boldsymbol{\eta})^{\mathrm{T}}. \tag{D 4b}$$

Specifically, on the right-hand side of (D 4*b*) we have $\bullet = \mathbf{G}^\ell = (\mathbf{G}^\ell)^T$.

As in (3.21), we employ the Taylor expansion

$$\mathbf{G}^\ell(\boldsymbol{x}, 1) = \exp(\mathrm{L}_\eta)\,\mathbf{G}^\ell(\boldsymbol{x}, 0). \tag{D 5}$$

Use of (D 2*b*,*c*) and (D 4*a*,*b*) gives

$$\mathbf{G}(\boldsymbol{x}) = \exp(\mathrm{L}_\eta)\boldsymbol{I} = \boldsymbol{I} + (\nabla\boldsymbol{\eta}) + (\nabla\boldsymbol{\eta})^{\mathrm{T}} + \tfrac{1}{2}\{\boldsymbol{\eta}\,,\ ((\nabla\boldsymbol{\eta}) + (\nabla\boldsymbol{\eta})^{\mathrm{T}})\} + \cdots, \tag{D 6}$$

which are needed to determine the linear relations

$$\boldsymbol{V} = \mathbf{G}\cdot\boldsymbol{v}, \qquad \boldsymbol{W} = \mathbf{G}\cdot\boldsymbol{w}, \qquad \boldsymbol{U} = \mathbf{G}\cdot\boldsymbol{u} \tag{D 7a,b,c}$$

between the covariant (e.g. $\boldsymbol{V}$, see (2.3*a*)) and contravariant (e.g. $\boldsymbol{v}$, see (2.3*b*)) velocity vectors, where

$$\boldsymbol{V} = (\nabla\boldsymbol{x}^L)\cdot\boldsymbol{v}^{*L}, \qquad \boldsymbol{W} = (\nabla\boldsymbol{x}^L)\cdot\boldsymbol{w}^{*L}, \qquad \boldsymbol{u} = (\nabla\boldsymbol{x}^L)\cdot\boldsymbol{u}^{*L}, \tag{D 7d,e,f}$$

$$\boldsymbol{v}^{*L} = \boldsymbol{v}\cdot\nabla\boldsymbol{x}^L, \qquad \boldsymbol{w}^{*L} = \boldsymbol{w}\cdot\nabla\boldsymbol{x}^L, \qquad \boldsymbol{u}^{*L} = \boldsymbol{u}\cdot\nabla\boldsymbol{x}^L, \tag{D 7g,h,i}$$

and $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w}$ (see (2.8*c*)).

Appendix E. Formula (3.30*b*) for $\boldsymbol{w}^*$

The Lie dragging contravariant velocity vector $\boldsymbol{\eta}$ has the mathematically obvious property that its Lie derivative vanishes: $\mathrm{L}_\eta\,\boldsymbol{\eta} = [\boldsymbol{\eta},\ \boldsymbol{\eta}] = \mathbf{0}$. It implies that its dragged-along value $\boldsymbol{\eta}^\ell(\boldsymbol{x}, \tau)$ (3.3*b*) remains unchanged equal to $\boldsymbol{\eta}(\boldsymbol{x})$ (3.11*b*) and actually takes the Eulerian value $\boldsymbol{\eta}^*(\boldsymbol{x})$ (3.11*c*) (see also Appendix A). This property leads to the non-trivial consequence (3.11*d*) or more explicitly (5.2*a*).

In the same spirit, $\partial_t\boldsymbol{\eta}$ is a contravariant vector with Lie derivative

$$\mathrm{L}_\eta\partial_t\boldsymbol{\eta} = [\boldsymbol{\eta},\ \partial_t\boldsymbol{\eta}], \tag{E 1}$$

which enables us to express (3.30*a*) in the operator form

$$\boldsymbol{w}(\boldsymbol{x}, t) = \mathrm{L}_\eta^{-1}(\exp(\mathrm{L}_\eta)\ \partial_t\boldsymbol{\eta} - \partial_t\boldsymbol{\eta}). \tag{E 2a}$$

Being a contravariant vector, the velocity $\boldsymbol{w}$ obeys the same transformation law (3.22*c*) as $\boldsymbol{v}$ and so has the inverse $\boldsymbol{w}^* = \exp(-\mathrm{L}_\eta)\,\boldsymbol{w}$. Combined with (E 2*a*), it determines

$$\boldsymbol{w}^*(\boldsymbol{x}, t) = (-\mathrm{L}_\eta)^{-1}(\exp(-\mathrm{L}_\eta)\partial_t\boldsymbol{\eta} - \partial_t\boldsymbol{\eta}). \tag{E 2b}$$

Just as (3.30*a*) is the expansion of (E 2*a*), (3.30*b*) is the expansion of (E 2*b*).

Appendix F. Modified Eulerian mean: a glm variant

The MEM equation presented by Roberts & Soward (2009), but obtained previously by Soward & Roberts (2008, equations (3.48*a*,*b*)), is

$$\partial_t \overline{\boldsymbol{V}^E} + \{\,\overline{\boldsymbol{v}^*},\, \overline{\boldsymbol{V}^E}\} + \overline{\{\boldsymbol{\zeta},\, \{\boldsymbol{v}^{*\prime},\, \boldsymbol{v}^*\}\}} = -\nabla \overline{\Pi^E}, \tag{F 1}$$

where $\boldsymbol{\zeta}$ solves

$$\boldsymbol{v}^{*\prime} = \partial_t \boldsymbol{\zeta} + [\,\overline{\boldsymbol{v}^*},\, \boldsymbol{\zeta}], \quad \nabla\cdot\boldsymbol{\zeta} = 0, \quad \overline{\boldsymbol{\zeta}} = \boldsymbol{0}, \tag{F 2a,b,c}$$

and

$$\overline{\boldsymbol{V}^E} = \overline{\boldsymbol{v}^*} + \overline{\{\boldsymbol{\zeta},\, \boldsymbol{v}^{*\prime}\}}, \quad \overline{\Pi^E} = \overline{\Pi^*} + \overline{\boldsymbol{\zeta}\cdot\nabla\Pi^{*\prime}}. \tag{F 2d,e}$$

From (F 1) we may deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\overline{\mathscr{C}^*}} \overline{\boldsymbol{v}^*}\cdot\mathrm{d}\boldsymbol{x} = -\oint_{\overline{\mathscr{C}^*}} \overline{\{\boldsymbol{\zeta},\, \{\boldsymbol{v}^{*\prime},\, \boldsymbol{v}^*\}\}}\cdot\mathrm{d}\boldsymbol{x}, \tag{F 3}$$

for circuits $\overline{\mathscr{C}^*}(t)$ composed of points $\boldsymbol{x}$ moving with the mean velocity $\overline{\boldsymbol{v}^*}(\boldsymbol{x}, t)$. The MEM–glm equation (F 1), describing the advection of $\overline{\boldsymbol{V}^E}$ by $\overline{\boldsymbol{v}^*}$, may be regarded as an attempt to reproduce the structure of the GLM equation (4.4*a*), describing the advection of $\overline{\boldsymbol{V}}$ by $\boldsymbol{u}$, but it fails to do so because of the presence of the term $\overline{\{\boldsymbol{\zeta},\, \{\boldsymbol{v}^{*\prime},\, \boldsymbol{v}^*\}\}}$. If this term were absent as Holm (2002) claimed, then the right-hand side of (F 3) would vanish and a mean Kelvin's circulation theorem, analogous to the GLM version (4.4*c*), would follow. However, since the term $\overline{\{\boldsymbol{\zeta},\, \{\boldsymbol{v}^{*\prime},\, \boldsymbol{v}^*\}\}}$ is $O(\zeta^2)$ $(=O(\eta^2)$, see below), the proposed Kelvin's theorem does not hold to this order of accuracy sought by Holm (2002). Quite simply, for Kelvin's theorem to hold it is essential that the circuit involved is a material circuit composed of fluid particles. This is not the case for $\overline{\mathscr{C}^*}(t)$ in (F 3), whereas it is for $\mathscr{C}(t)$ in (4.4*c*).

Evidently, comparison of (4.12*a*) with (F 2*a*) shows that

$$\boldsymbol{\zeta} \approx \boldsymbol{\eta} \qquad (\eta \ll 1), \tag{F 4}$$

while (4.12*b*) may be re-expressed as

$$\boldsymbol{u} - \overline{\boldsymbol{v}^*} \approx -\tfrac{1}{2}\overline{[\boldsymbol{v}^{*\prime},\, \boldsymbol{\eta}]}, \quad \boldsymbol{v}^{*\prime} \approx \partial_t \boldsymbol{\eta} + [\,\overline{\boldsymbol{v}^*},\, \boldsymbol{\eta}], \tag{F 5a,b}$$

where the symbol $\approx$ continues to mean, as introduced in § 4.2, that only fluctuating terms correct to $O(\eta)$ and mean terms correct to $O(\eta^2)$ are retained. It follows from (4.13) that $\overline{\boldsymbol{V}^E}$ and $\overline{\Pi^E}$ may be expressed in alternative forms that satisfy

$$\overline{\boldsymbol{V}} - \overline{\boldsymbol{V}^E} \approx \tfrac{1}{2}\overline{\{\boldsymbol{\eta},\, \{\boldsymbol{\eta},\, \overline{\boldsymbol{v}^*}\}\}}, \quad \overline{\Pi} - \overline{\Pi^E} \approx \tfrac{1}{2}\overline{(\boldsymbol{\eta}\cdot\nabla)^2\overline{\Pi^*}}. \tag{F 6a,b}$$

The relations (F 5) and (F 6) enable us to see more clearly how the ETHEL–glm system (4.4*a*), (4.8), (4.12) and (4.13) is approximated, via (F 4), by the MEM–glm system (F 1) and (F 2). Although $\boldsymbol{u}$, $\overline{\boldsymbol{V}}$ and $\overline{\Pi}$ differ from $\overline{\boldsymbol{v}^*}$, $\overline{\boldsymbol{V}^E}$ and $\overline{\Pi^*}$, respectively, at $O(\eta^2)$, the GLM equation (4.4*a*) and the MEM–glm equation (F 1) must be consistent. To check this consistency correct to $O(\eta^2)$, we first recall that $\boldsymbol{v}^{*\prime} = O(\eta)$ implies that $\boldsymbol{v}^* = \overline{\boldsymbol{v}^*} + O(\eta)$. Then use of the approximate expressions (F 5), (F 6*a*,*b*) for the $O(\eta^2)$ differences $\boldsymbol{u} - \overline{\boldsymbol{v}^*}$, $\overline{\boldsymbol{V}} - \overline{\boldsymbol{V}^E}$ and $\overline{\Pi} - \overline{\Pi^E}$, respectively, shows that the difference of (4.4*a*) and (F 1) is

$$\begin{aligned}\tfrac{1}{2}\,\partial_t\overline{\{\boldsymbol{\eta},\, \{\boldsymbol{\eta},\, \overline{\boldsymbol{v}^*}\}\}} + \tfrac{1}{2}\{\,\overline{\boldsymbol{v}^*},\, \overline{\{\boldsymbol{\eta},\, \{\boldsymbol{\eta},\, \overline{\boldsymbol{v}^*}\}\}}\} - \tfrac{1}{2}\{\,\overline{[\boldsymbol{v}^{*\prime},\, \boldsymbol{\eta}]},\, \overline{\boldsymbol{v}^*}\}&\\ - \overline{\{\boldsymbol{\eta},\, \{\boldsymbol{v}^{*\prime},\, \overline{\boldsymbol{v}^*}\}\}} + \tfrac{1}{2}\nabla\,\overline{(\boldsymbol{\eta}\cdot\nabla)^2\overline{\Pi^*}} &\approx \boldsymbol{0}.\end{aligned} \tag{F 7}$$

Direct differentiation of $\overline{\{\boldsymbol{\eta},\ \{\boldsymbol{\eta},\ \overline{\boldsymbol{v}^*}\}\}}$ with respect to t yields

$$\partial_t\overline{\{\boldsymbol{\eta},\ \{\boldsymbol{\eta},\ \overline{\boldsymbol{v}^*}\}\}} = \overline{\{\partial_t\boldsymbol{\eta},\ \{\boldsymbol{\eta},\ \overline{\boldsymbol{v}^*}\}\}} + \overline{\{\boldsymbol{\eta},\ \{\partial_t\boldsymbol{\eta},\ \overline{\boldsymbol{v}^*}\}\}} + \overline{\{\boldsymbol{\eta},\ \{\boldsymbol{\eta},\ \partial_t\overline{\boldsymbol{v}^*}\}\}}, \tag{F 8}$$

in which we substitute $\partial_t\overline{\boldsymbol{v}^*} = -\{\overline{\boldsymbol{v}^*}, \overline{\boldsymbol{v}^*}\} - \nabla\overline{\Pi^*} + O(\eta)$ from the average of (1.1d) and $\partial_t\boldsymbol{\eta} = \boldsymbol{v}^{*\prime} - [\overline{\boldsymbol{v}^*},\ \boldsymbol{\eta}] + O(\eta^2)$ from (F 5b). Repeated use of the commutator properties

$$\{\boldsymbol{\eta},\ \nabla\overline{\Pi^*}\} = \nabla(\boldsymbol{\eta}\cdot\nabla\overline{\Pi^*}), \tag{F 9a}$$

$$\{[\boldsymbol{v}^{*\prime},\ \boldsymbol{\eta}],\ \overline{\boldsymbol{v}^*}\} = \{\boldsymbol{v}^{*\prime},\ \{\boldsymbol{\eta},\ \overline{\boldsymbol{v}^*}\}\} - \{\boldsymbol{\eta},\ \{\boldsymbol{v}^{*\prime},\ \overline{\boldsymbol{v}^*}\}\}, \tag{F 9b}$$

valid for arbitrary vectors $\boldsymbol{v}^{*\prime}$, $\overline{\boldsymbol{v}^*}$, $\boldsymbol{\eta}$ and scalar $\overline{\Pi^*}$, then verifies the validity of (F 7), which provides the check of the consistency of (4.4a) and (F 1) that we sought.

The Taylor hypothesis of Holm (2002) and the additional simplifying assumptions of § 4.3 lead to the results

$$\overline{\boldsymbol{V}^E} \approx \overline{\boldsymbol{V}} - \tfrac{1}{2}\alpha^2\nabla^2\overline{\boldsymbol{v}^*} = \overline{\boldsymbol{v}^*} - \alpha^2\nabla^2\overline{\boldsymbol{v}^*}, \tag{F 10a}$$

$$\begin{aligned}\overline{\{\boldsymbol{\zeta},\ \{\boldsymbol{v}^{*\prime},\ \boldsymbol{v}^*\}\}} &\approx -\alpha^2\nabla_j\{\nabla_j\overline{\boldsymbol{v}^*},\ \overline{\boldsymbol{v}^*}\}\\ &= -\tfrac{1}{2}\alpha^2(\nabla^2\{\overline{\boldsymbol{v}^*},\ \overline{\boldsymbol{v}^*}\} + \{\nabla^2\overline{\boldsymbol{v}^*},\ \overline{\boldsymbol{v}^*}\} - \{\overline{\boldsymbol{v}^*},\ \nabla^2\overline{\boldsymbol{v}^*}\}).\end{aligned} \tag{F 10b}$$

Substitution into (F 1) leads to the MEM-based equation (4.19a).

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